Classical Electrodynamics

Chapter 5

Magnetostatics, Faraday's Law,
Quasi-Static Fields

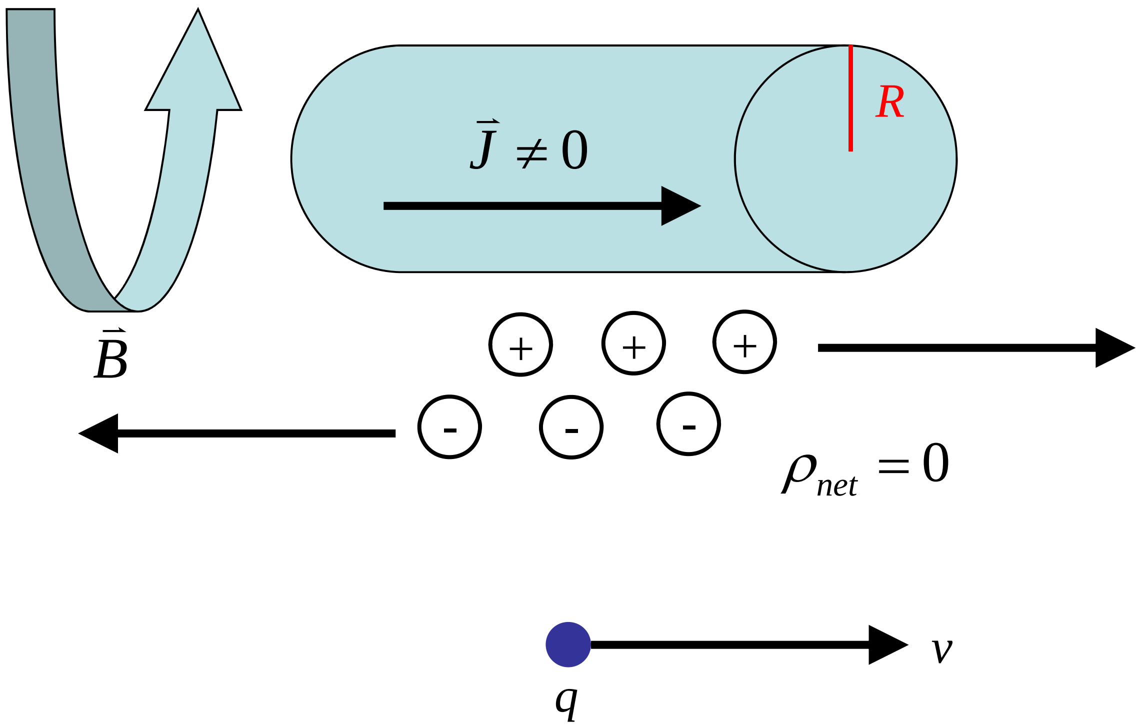
Contents

§5. 1 The relationship between electric field and magnetic field

(1) The force on a charge acted by a nearby conduction wire is:

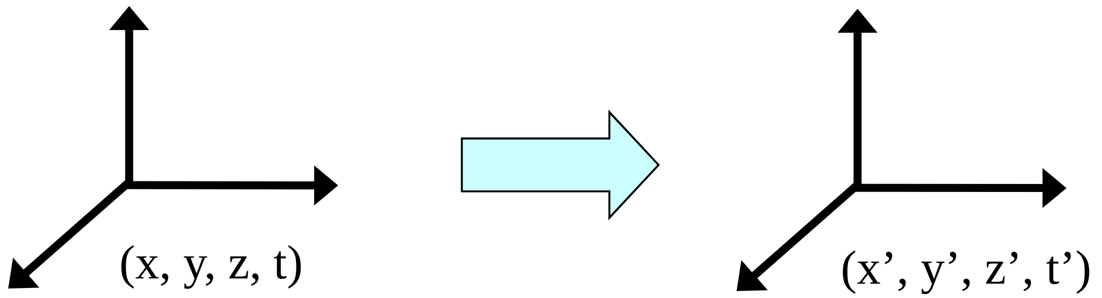
In the inertial coordinate, the charge experiences a magnetic force: $\vec{F} = q\vec{v} \times \vec{B}$

In the relative coordinate, where the observation is performed in the coordinate of charge, it experiences a electric force: $\vec{F} = q\vec{E}$



§5. 1 The relationship between electric field and magnetic field

(2)



With the transformation of coordinates in the special relativity:

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \; y' = y, \; z' = z, \; t' = \frac{t - v/c^2\, x}{\sqrt{1 - v^2/c^2}}, \; \rho' = -\frac{v}{c^2}\frac{j_x}{\sqrt{1 - v^2/c^2}}$$

With the Gauss's law: $E \cdot 2\pi r \cdot l = \dfrac{\rho' \pi R^2 l}{\varepsilon_0}$

In the relative coordinate:

$$F = qE = q\frac{\pi R^2 j_x}{2\pi r \varepsilon_0}\frac{v}{c}\frac{1}{\sqrt{1 - v^2/c^2}} = q\frac{\mu_0 I}{2\pi r} v \frac{1}{\sqrt{1 - v^2/c^2}}$$

§5. 1 The relationship between electric field and magnetic field

But when we consider the situation in the inertial coordinate:

$$F = q\frac{\mu_0 I}{2\pi r}v = qvB = \left|q\vec{v}\times\vec{B}\right|$$

This means that by proper transformation of the coordinate, we can just deal with the electric force to solve the electromagnetic problems. Otherwise, the concept of the magnetic force must be introduced. In general, the force can be expressed as:

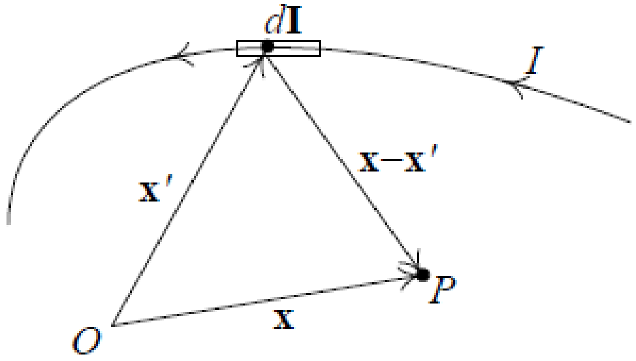
$$\vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

§5.2 Biot and Savart law and vetor potential

(1) The magnetic field generated by a short segment of wire carrying current is given by:

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{Id\vec{l}\,' \times (\vec{x} - \vec{x}\,')}{|\vec{x} - \vec{x}\,'|^3}$$

$$\Rightarrow \vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}\,') \times (\vec{x} - \vec{x}\,')}{|\vec{x} - \vec{x}\,'|^3} d^3x'$$



(2) $\because \vec{\nabla} \times (f\vec{A}) = \vec{\nabla} f \times \vec{A} + f(\vec{\nabla} \times \vec{A})$

$$\therefore \vec{B} = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}\,') \times \vec{\nabla}\left(\frac{-1}{|\vec{x} - \vec{x}\,'|}\right) d^3x'$$

$$= \frac{\mu_0}{4\pi} \int \vec{\nabla} \times \left(\frac{\vec{J}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|}\right) d^3x' - \frac{\mu_0}{4\pi} \int \frac{1}{|\vec{x} - \vec{x}\,'|} \vec{\nabla} \times \vec{J}(\vec{x}\,') d^3x' = \vec{\nabla} \times \vec{A}$$

Where the vector potential is: $\vec{A} = \dfrac{\mu_0}{4\pi} \displaystyle\int \frac{\vec{J}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|} d^3x'$

§5. 3 Differential equations of magnetostatics and Ampere's law

(1) $\because \vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow \vec{\nabla} \bullet \vec{B} = \vec{\nabla} \bullet \left(\vec{\nabla} \times \vec{A}\right) = 0$

(2) $\vec{\nabla} \times \vec{B} = \vec{\nabla} \times \vec{\nabla} \times \left[\frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x' \right]$

$\left(\because \vec{\nabla} \times \vec{\nabla} \times \vec{c} = \vec{\nabla}\left(\vec{\nabla} \bullet \vec{c}\right) - \vec{\nabla}^2 \vec{c}\right)$

$$= \frac{\mu_0}{4\pi} \left\{ \int \vec{\nabla} \left[\vec{\nabla} \bullet \left(\frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} \right) \right] d^3x' - \int \vec{\nabla}^2 \left(\frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} \right) d^3x' \right\}$$

$\left(\because \vec{\nabla} \bullet \left(f\vec{A}\right) = \vec{\nabla} f \bullet \vec{A} + f\left(\vec{\nabla} \bullet \vec{A}\right)\right)$

$$= \frac{\mu_0}{4\pi} \vec{\nabla} \left\{ \int \vec{\nabla} \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) \bullet \vec{J}(\vec{x}') d^3x' + \int \frac{1}{|\vec{x} - \vec{x}'|} \vec{\nabla} \bullet \vec{J}(\vec{x}') d^3x' \right\}$$

$$- \frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}') \vec{\nabla}^2 \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) d^3x'$$

§5. 3 Differential equations of magnetostatics and Ampere's law

$$\left(\because \vec{\nabla}^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -4\pi\delta(\vec{x}-\vec{x}')\right)$$

$$= \frac{\mu_0}{4\pi}\vec{\nabla}\int\vec{\nabla}'\left(\frac{-1}{|\vec{x}-\vec{x}'|}\right)\bullet\vec{J}(\vec{x}')d^3x' + \mu_0\vec{J}(\vec{x})$$

$$\left(\because \vec{\nabla}'\bullet\left(f\vec{A}\right) = \vec{\nabla}' f \bullet \vec{A} + f\left(\vec{\nabla}'\bullet\vec{A}\right)\right)$$

$$= -\frac{\mu_0}{4\pi}\vec{\nabla}\int\vec{\nabla}'\bullet\left(\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right)d^3x' + \frac{\mu_0}{4\pi}\vec{\nabla}\int\frac{\vec{\nabla}'\bullet\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x' + \mu_0\vec{J}(\vec{x})$$

$$\left(\because \int\vec{\nabla}'\bullet\left(\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right)d^3x' = \oint\left(\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right)da' \to 0, \text{as } S \to \infty\right)$$

$$= \frac{\mu_0}{4\pi}\vec{\nabla}\int\frac{\vec{\nabla}'\bullet\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x' + \mu_0\vec{J}(\vec{x})$$

§5. 3 Differential equations of magnetostatics and Ampere's law

For static electromagnetics, we get the Ampere's law:

$$\vec{\nabla} \bullet \vec{J} = 0 \Rightarrow \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

Generally, the Ampere's law should be modified as the Maxwell-Ampere's law:

$$\vec{\nabla} \bullet \vec{J} = -\frac{\partial \rho}{\partial t}$$

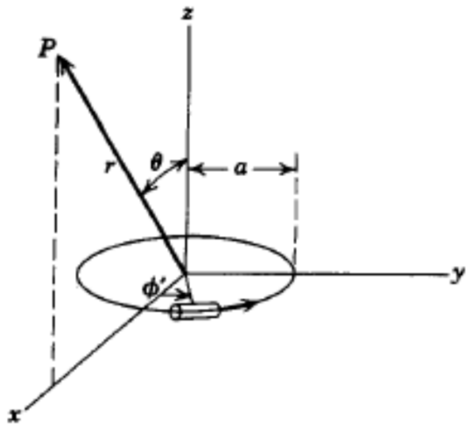
$$\therefore \vec{\nabla} \times \vec{B} = \frac{\mu_0}{4\pi} \vec{\nabla} \int \frac{-\dfrac{\partial \rho(\vec{x}')}{\partial t}}{|\vec{x} - \vec{x}'|} d^3x' + \mu_0 \vec{J}$$

$$= \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \left[-\vec{\nabla} \left(\frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x' \right) \right]$$

$$= \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

§5.4 Vector potential and magnetic induction for a circular current loop

(1) $\vec{A}(\vec{x}) = \dfrac{\mu_0 I}{4\pi}\displaystyle\int \dfrac{d\vec{l}\,'}{|\vec{x}-\vec{x}'|}$



$$d\vec{l}\,' = a d\phi' \hat{a}_\phi = -a\sin\phi' d\phi' \hat{a}_x + a\cos\phi' d\phi' \hat{a}_y$$

$$\because |\vec{x}-\vec{x}'| = \sqrt{(x-a\cos\phi')^2 + (y-a\sin\phi')^2 + (z-0)^2}$$

$$= \sqrt{r^2 + a^2 - 2ra\sin\theta\cos(\phi'-\phi)}$$

$$\therefore A_x(\vec{x}) = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{-a\sin\phi' d\phi'}{\sqrt{r^2 + a^2 - 2ra\sin\theta\cos(\phi'-\phi)}}$$

$$A_y(\vec{x}) = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{a\cos\phi' d\phi'}{\sqrt{r^2 + a^2 - 2ra\sin\theta\cos(\phi'-\phi)}}$$

$$\because \hat{a}_x = \sin\theta\cos\phi\hat{a}_r + \cos\theta\cos\phi\hat{a}_\theta - \sin\phi\hat{a}_\phi$$

$$\hat{a}_y = \sin\theta\sin\phi\hat{a}_r + \cos\theta\sin\phi\hat{a}_\theta + \cos\phi\hat{a}_\phi$$

$$\hat{a}_z = \cos\theta\hat{a}_r - \sin\theta\hat{a}_\theta$$

§5.4 Vector potential and magnetic induction for a circular current loop

$$\therefore A_r = \frac{\mu_0 I a}{4\pi} \int_0^{2\pi} \frac{\sin\theta \sin(\phi-\phi')d\phi'}{\sqrt{r^2+a^2-2ra\sin\theta\cos(\phi'-\phi)}}$$

$$= \frac{\mu_0 I a}{4\pi}\frac{1}{ra}\sqrt{r^2+a^2-2ra\sin\theta\cos(\phi-\phi')}\Big|_0^{2\pi} = 0$$

$$A_\theta = \frac{\mu_0 I a}{4\pi} \int_0^{2\pi} \frac{\cos\theta \sin(\phi-\phi')d\phi'}{\sqrt{r^2+a^2-2ra\sin\theta\cos(\phi'-\phi)}}$$

$$= \frac{\mu_0 I a}{4\pi}\frac{\cos\theta}{\sin\theta}\frac{1}{ra}\sqrt{r^2+a^2-2ra\sin\theta\cos(\phi-\phi')}\Big|_0^{2\pi} = 0$$

$$A_\phi = \frac{\mu_0 I a}{4\pi} \int_0^{2\pi} \frac{\cos(\phi-\phi')d\phi'}{\sqrt{r^2+a^2-2ra\sin\theta\cos(\phi'-\phi)}} \neq 0$$

§5.4 Vector potential and magnetic induction for a circular current loop

Expand the denominator of A_ϕ with binomial expansion:

$$A_\phi = \frac{\mu_0 I a}{4\pi}\int_0^{2\pi}\frac{\cos(\phi'-\phi)}{\sqrt{r^2+a^2}}\left[1-\frac{2ra}{r^2+a^2}\sin\theta\cos(\phi'-\phi)\right]^{-1/2} d\phi'$$

$$= \frac{\mu_0 I a}{4\pi\sqrt{r^2+a^2}}\left\{\int_0^{2\pi}\cos(\phi'-\phi)d\phi' + \int_0^{2\pi}\left(-\frac{1}{2}\right)\frac{1}{1!}\left(-\frac{2ra\sin\theta}{r^2+a^2}\right)\cos^2(\phi'-\phi)d\phi'\right.$$

$$+\int_0^{2\pi}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{1}{2!}\left(-\frac{2ra\sin\theta}{r^2+a^2}\right)^2\cos^3(\phi'-\phi)d\phi'$$

$$\left.+\int_0^{2\pi}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\frac{1}{3!}\left(-\frac{2ra\sin\theta}{r^2+a^2}\right)^3\cos^4(\phi'-\phi)d\phi' + \ldots\ldots\right\}$$

$$\left(\text{Note} \int_0^{2\pi}\cos^4\theta d\theta = \int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta = \int_0^{2\pi}\frac{1+2\cos 2\theta+\cos^2 2\theta}{4}d\theta\right)$$

$$= \frac{\mu_0 I a^2 \pi r\sin\theta}{4\pi(r^2+a^2)^{3/2}}\left[0+1+0+\frac{15}{8}\left(\frac{ra\sin\theta}{r^2+a^2}\right)^2+\ldots\ldots\right]$$

§5.4 Vector potential and magnetic induction for a circular current loop

$$\text{let } m = I\pi a^2 \text{ \& assume } r >> a \Rightarrow A_\phi \cong \frac{\mu_0 m r \sin\theta}{4\pi r^3} = \frac{\mu_0 \vec{m}\times\vec{r}}{4\pi r^3}$$

(2) $$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{a}_1 & h_2\hat{a}_2 & h_3\hat{a}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta\hat{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ 0 & 0 & r\sin\theta A_\phi \end{vmatrix}$$

$$\therefore B_\theta = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\mu_0 m r\sin\theta}{4\pi r^3}\right) = \frac{\mu_0 m \sin\theta}{4\pi r^3}$$

$$B_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\mu_0 m r\sin\theta}{4\pi r^3}\right) = \frac{\mu_0 m 2\cos\theta}{4\pi r^3}$$

$$\Rightarrow \vec{B} = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\hat{a}_r + \sin\theta\hat{a}_\theta\right)$$

§5.5 Analogy between electric dipole and magnetic dipole

(1) Electric dipole: $$\Phi = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\bullet(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

$$\vec{E} = -\vec{\nabla}\Phi = \frac{1}{4\pi\varepsilon_0}\frac{3\hat{n}(\vec{p}\bullet\hat{n})-\vec{p}}{|\vec{x}-\vec{x}'|^3}$$

(2) magnetic dipole: $$\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\bullet(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{\mu_0}{4\pi}\frac{3\hat{n}(\vec{m}\bullet\hat{n})-\vec{m}}{|\vec{x}-\vec{x}'|^3}$$

§5. 6 Magnetic scalar potential

(1) If $\vec{J}=0 \Rightarrow \vec{\nabla}\times\vec{H}=0$, in free space : $\vec{B}=-\mu_0\vec{\nabla}\Phi_M$

To illustrate this concept, we consider the problem as follows. For the magnetic induction at the point P with coordinate $\vec{x}$ produced by an increment of current $Id\vec{l}\,'$at $\vec{x}\,'$, the magnetic induction can be explicitly expressed as:

$$d\Phi_M=\left(\vec{\nabla}\Phi_M\right)\bullet d\vec{x}=-\frac{\vec{B}\bullet d\vec{x}}{\mu_0}=-\left(\frac{I}{4\pi}\oint\frac{d\vec{l}\times(\vec{x}-\vec{x}\,')}{\left|\vec{x}-\vec{x}\,'\right|^3}\right)\bullet d\vec{x}$$

$$=-\frac{I}{4\pi}\oint\frac{d\vec{l}\times\vec{R}}{R^3}\bullet d\vec{x}=-\frac{I}{4\pi}d\Omega$$

where the solid angle is:

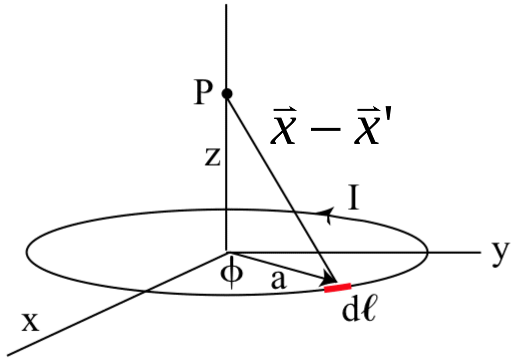
$$\Omega=\int\frac{d\vec{A}\bullet(-\vec{R})}{R^3}$$

$$\Rightarrow\Phi_M=-\frac{I\Omega}{4\pi}\Rightarrow\vec{B}=-\mu_0\vec{\nabla}\Phi_M=\frac{\mu_0 I}{4\pi}\vec{\nabla}\Omega$$

§5. 6 Magnetic scalar potential

(2) As an example, find the magnetic induction at a point on the z-axis:

(i) Directly find magnetic induction:



$$\vec{J}(\vec{x}') = I\delta(\rho'-a)\delta(z')\hat{a}_\phi$$

$$\vec{x}-\vec{x}' = z\hat{a}_z - a\hat{a}_\rho, \left|\vec{x}-\vec{x}'\right| = \sqrt{z^2+a^2}$$

$$\therefore \vec{J}(\vec{x}')\times\left(\vec{x}-\vec{x}'\right) = I\delta(\rho'-a)\delta(z')\left(z\hat{a}_\rho + a\hat{a}_z\right)$$

$$\therefore \vec{B} = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{x}')\times\left(\vec{x}-\vec{x}'\right)}{\left|\vec{x}-\vec{x}'\right|^3}\rho'd\rho'd\phi'dz' = \frac{\mu_0 I}{4\pi}\int_0^{2\pi}\frac{0\hat{a}_\rho + a\hat{a}_z}{\left(z^2+a^2\right)^{3/2}}ad\phi'$$

$$= \frac{\mu_0 I}{2}\frac{a\hat{a}_z}{\left(z^2+a^2\right)^{3/2}}$$

§5. 6 Magnetic scalar potential

(ii) With the concept of the magnetic scalar potential:

$$\Omega = \int \frac{d\vec{A} \bullet (-\vec{R})}{R^3}, \vec{R} = z\hat{a}_z - \rho' \hat{a}_\phi$$

$$\because \vec{R} \bullet d\vec{A} = z\rho' d\rho' d\phi'$$

$$\therefore -\Omega = \int_0^a \int_0^{2\pi} \frac{z\rho' d\rho' d\phi'}{\left(z^2 + \rho'^2\right)^{3/2}} = 2\pi\left[1 - \frac{z}{\sqrt{z^2 + a^2}}\right]$$

$$\therefore \Phi_M = -\frac{I\Omega}{4\pi} = -\frac{I}{2}\left[\frac{z}{\sqrt{z^2 + a^2}} - 1\right]$$

$$\Rightarrow B_z = -\mu_0 \frac{\partial \Phi_M}{\partial z} = \frac{\mu_0 I}{2} \frac{a^2}{\left(z^2 + a^2\right)^{3/2}}$$

§5. 6 Magnetic scalar potential

(iii) Since this problem has the property of ϕ symmetry, we can expand the magnetic scalar potential with the help of the Legendre polynomial. Besides, the magnetic scalar potential at any point can be obtained with the knowing the magnetic scalar potential on the z-axis:

(a) For $r < a$: $\Phi_M = \sum_l A_l r^l P_l(\cos\theta)$

$$\theta = 0 \Rightarrow r = z,\ P_l(1) = 1 \Rightarrow \Phi_M(z) = \sum_l A_l z^l$$

Expand $\Phi_M(z)$ with the binomial expansion, and note that the constant term of the $\Phi_M(z)$ can be dropped without loss:

$$\Phi_M(z) = -\frac{Iz}{2a}\left[1+\left(\frac{z}{a}\right)^2\right]^{-1/2}$$

$$= -\frac{Iz}{2a}\left[1 - \frac{1}{2}\frac{1}{1!}\left(\frac{z}{a}\right)^2 + \frac{1}{2}\frac{3}{2}\frac{1}{2!}\left(\frac{z}{a}\right)^4 - \frac{1}{2}\frac{3}{2}\frac{5}{2}\frac{1}{3!}\left(\frac{z}{a}\right)^6 + \ldots\ldots\right]$$

§5. 6 Magnetic scalar potential

$$= -\frac{Iz}{2a} + \frac{Iz^3}{4a^3} - \frac{3Iz^5}{16a^5} + \frac{5Iz^7}{32a^7} + ...$$

$$\therefore A_1 = -\frac{I}{2a}, A_2 = \frac{I}{4a^3}, A_1 = -\frac{3I}{16a^5}, A_1 = \frac{5I}{32a^7}, ...$$

$$\therefore \Phi_M(r,\theta) = -\frac{I}{2}\left[\frac{r}{a}P_1(\cos\theta) - \frac{r^3}{2a^3}P_3(\cos\theta) + \frac{3r^5}{8a^5}P_5(\cos\theta) - \frac{5r^7}{16a^7}P_7(\cos\theta) + ...\right]$$

§5. 6 Magnetic scalar potential

(b) For $r > a$: $\Phi_M(z) = A_0 + \sum_l B_l \frac{1}{z^{l+1}}$

$$\theta = 0 \Rightarrow r = z, P_l(1) = 1 \Rightarrow \Phi_M(z) = \sum_l A_l z^l$$

Expand $\Phi_M(z)$ with the binomial expansion, and note that the constant term of the $\Phi_M(z)$ can be dropped without loss:

$$\Phi_M(z) = -\frac{I}{2}\left[1+\left(\frac{a}{z}\right)^2\right]^{-1/2}$$

$$= -\frac{I}{2}\left[1-\frac{1}{2}\frac{1}{1!}\left(\frac{a}{z}\right)^2+\frac{1}{2}\frac{3}{2}\frac{1}{2!}\left(\frac{a}{z}\right)^4-\frac{1}{2}\frac{3}{2}\frac{5}{2}\frac{1}{3!}\left(\frac{a}{z}\right)^6+\ldots\ldots\right]$$

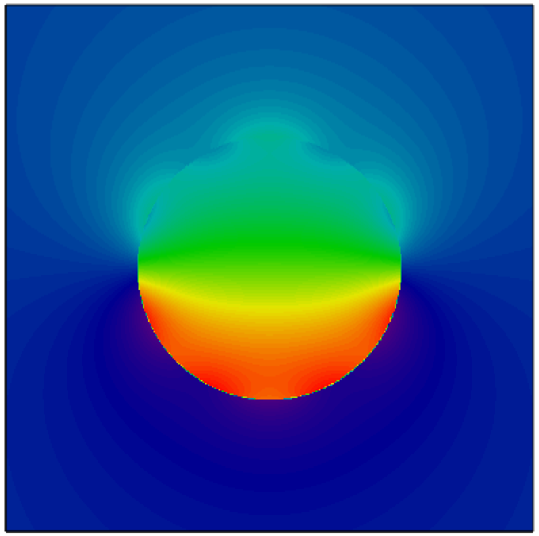
$$= -\frac{I}{2}+\frac{Ia^2}{4z^2}-\frac{3Ia^4}{16z^4}+\frac{5Ia^6}{32z^6}+\ldots\ldots$$

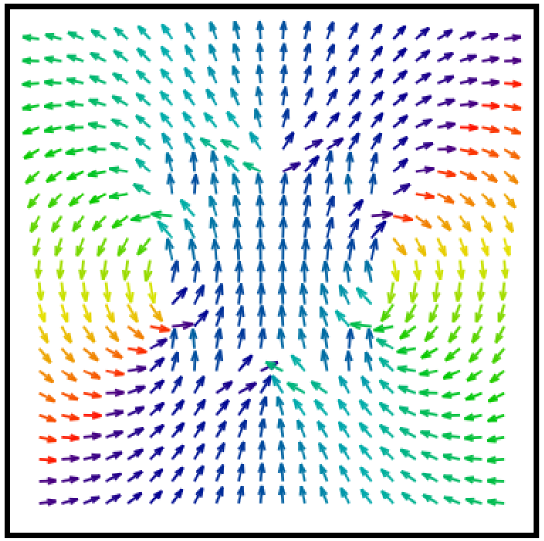
$$\therefore A_0 = -\frac{I}{2}, B_1 = \frac{I}{4}a^2, B_3 = -\frac{3I}{16}a^4, B_5 = \frac{5I}{32}a^6, \ldots\ldots$$

§5. 6 Magnetic scalar potential

$$\therefore \Phi_M(r,\theta)=\frac{I}{2}\left[-1+\frac{a^2}{2r^2}P_1(\cos\theta)-\frac{3a^4}{8r^4}P_3(\cos\theta)\right.$$
$$\left.+\frac{5a^6}{16r^6}P_5(\cos\theta)+......\right]$$

The figure below shows the simulation of the magnetic scalar potential viewed from radial ρ axis with the parameters of I = 0.1 and a = 1:





§5.7 Magnetic moment

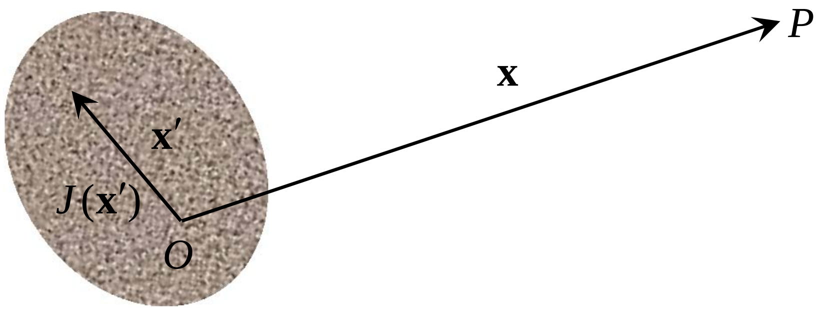
For a localized current density, we can use Taylor expansion:

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{|\vec{x}|} + \frac{\vec{x}\bullet\vec{x}'}{|\vec{x}|^3} + \ldots\ldots$$

$$\because \int \vec{J}(\vec{x}')d^3x' = I\oint d\vec{l}\,' = 0 \Rightarrow \text{ no magnetic monopole}$$

$$\therefore \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x' = \frac{\mu_0}{4\pi|\vec{x}|^3}\int \vec{J}(\vec{x}')(\vec{x}\bullet\vec{x}')d^3x' + \ldots\ldots$$

$$\therefore A_i(\vec{x}) = \frac{\mu_0\vec{x}\bullet}{4\pi|\vec{x}|^3}\int J_i(\vec{x}')\vec{x}'\,d^3x' + \ldots\ldots$$



§5.7 Magnetic moment

$$\because\ \vec{x}\bullet\int J_i(\vec{x}')\vec{x}'d^3x' = \sum_j x_j\int J_i x_j' d^3x'$$

In the text book, it is pointed out that: $\int\left(x_i' J_j + x_j' J_i\right)d^3x' = 0$

$$\therefore\ \vec{x}\bullet\int J_i(\vec{x}')\vec{x}'d^3x' = -\frac{1}{2}\sum_j x_j\int\left(x_i' J_j - x_j' J_i\right)d^3x'$$

$$= -\frac{1}{2}\left[\vec{x}\times\int\left(\vec{x}'\times\vec{J}(\vec{x}')\right)d^3x'\right]$$

It is customary to define the magnetic moment: $\vec{m} = \frac{1}{2}\int\left(\vec{x}'\times\vec{J}(\vec{x}')\right)d^3x'$

Consequently, the magnetic dipole vector potential is: $\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{x}}{|\vec{x}|^3}$

And the magnetic induction outside the localized source is: $\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{3\hat{n}(\vec{m}\bullet\hat{n})-\vec{m}}{|\vec{x}|^3}$

§ 5.8 Macroscopic equations, boundary conditions on $\vec{B}$ and $\vec{H}$

(1) $\vec{\nabla}\bullet\vec{B}=0\Rightarrow\vec{B}=\vec{\nabla}\times\vec{A}$

(2) Magnetization: $\vec{M}=\sum_i N_i\langle\vec{m}_i\rangle$

(3) With the bulk magnetization and a macrosopic current density:

$$\Delta\vec{A}(\vec{x})=\frac{\mu_0}{4\pi}\frac{\vec{J}(\vec{x}')\Delta V}{|\vec{x}-\vec{x}'|}+\frac{\mu_0}{4\pi}\frac{\vec{M}(\vec{x}')\times(\vec{x}-\vec{x}')\Delta V}{|\vec{x}-\vec{x}'|^3}$$

$$\Rightarrow\vec{A}(\vec{x})=\frac{\mu_0}{4\pi}\left[\int\left(\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}+\frac{\vec{M}(\vec{x}')\times(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\right)d^3x'\right]$$

$$=\frac{\mu_0}{4\pi}\left[\int\left(\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}+\vec{M}(\vec{x}')\times\vec{\nabla}'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\right)d^3x'\right]$$

$$\left(\because\vec{\nabla}'\times(f\vec{A})=\vec{\nabla}'f\times\vec{A}+(\vec{\nabla}'\times\vec{A})f\right)$$

$$\left(\because\int\vec{\nabla}'\times\left(\frac{\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right)d^3x'\to 0,\text{ as } S\to\infty\right)$$

§ 5.8 Macroscopic equations, boundary conditions on $\vec{B}$ and $\vec{H}$

$$= \frac{\mu_0}{4\pi}\left[\int\left(\frac{\vec{J}(\vec{x}') + \vec{\nabla}'\times\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right)d^3x'\right]$$

interpret: $\vec{\nabla}\times\vec{M}(\vec{x}) = \vec{J}_M$

$$\Rightarrow \vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J} + \vec{J}_M\right)$$

$$\Rightarrow \vec{\nabla}\times\vec{H} = \vec{\nabla}\times\left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = J$$

If the material is linear: $\vec{B} = \mu\vec{H} : \begin{cases} \mu > \mu_0 : \text{paramagnetic} \\ \mu < \mu_0 : \text{diamagnetic} \end{cases}$

(4) Boundary conditions: (the same discussion as for the electrostatics)

$\vec{\nabla}\bullet\vec{B} = 0 : B_{1n} = B_{2n}$

$\vec{\nabla}\times\vec{H} = \vec{J} : H_{2t} - H_{1t} = K$, where K is a surface current density

§ 5.9 Methods of solving boundary-value problems in magnetostatics

(1) Generally applicable method of the vector potential:

$$\because \vec{\nabla} \bullet \vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla} \times \vec{A}$$

$$\text{if } \vec{B} = \mu\vec{H}$$

$$\Rightarrow \vec{\nabla} \times \vec{H} = \frac{1}{\mu}\vec{\nabla} \times \vec{B} = \frac{1}{\mu}\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{J}$$

$$\because \vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}\left(\vec{\nabla} \bullet \vec{A}\right) - \vec{\nabla}^2 \vec{A}$$

$$\text{choose Coulomb gauge}: \vec{\nabla} \bullet \vec{A} = 0$$

$$\Rightarrow \vec{\nabla}^2 \vec{A} = -\mu\vec{J} : \text{Poisson equation}$$

(2) $\vec{J} = 0 \Rightarrow$ magnetic scalar potential

$$\because \vec{\nabla} \times \vec{H} = 0 \Rightarrow \vec{H} = -\vec{\nabla}\Phi_M$$

$$\text{if } \vec{B} = \mu\vec{H} \Rightarrow \vec{\nabla} \bullet \vec{B} = -\mu\vec{\nabla}^2\Phi_M = 0$$

$$\Rightarrow \vec{\nabla}^2\Phi_M = 0 : \text{Laplace equation}$$

(3) Hard ferromagnetic $\left(\vec{M} \text{ given}, \vec{J} = 0\right)$

$$\because \vec{J} = 0 \Rightarrow \vec{H} = -\vec{\nabla}\Phi_M$$

$$\vec{\nabla} \bullet \vec{B} = \mu_0 \vec{\nabla} \bullet \left(\vec{H} + \vec{M}\right) = 0, \text{ and define } \vec{\nabla} \bullet \vec{M} = -\rho_M$$

$$\Rightarrow \vec{\nabla}^2 \Phi_M = -\rho_M \text{ : Poisson equation}$$

(i) In free space and no surface contribution:

$$\Phi_M(\vec{x}) = \frac{1}{4\pi}\int \frac{\rho_M(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x' = -\frac{1}{4\pi}\int \frac{\vec{\nabla}' \bullet \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x'$$

(ii) With surface contribution:

$$\Phi_M(\vec{x}) = -\frac{1}{4\pi}\int \frac{\vec{\nabla}' \bullet \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x' + \frac{1}{4\pi}\oint \frac{\hat{n} \bullet \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} da'$$

§ 5.9 Methods of solving boundary-value problems in magnetostatics

Moreover:

$$\Phi_M(\vec{x}) = -\frac{1}{4\pi}\int \frac{\vec{\nabla}'\bullet\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x'$$

$$\left(\because \vec{\nabla}'\bullet(\vec{M}f) = f\left(\vec{\nabla}'\bullet\vec{M}\right) + \vec{\nabla}' f \bullet \vec{M}\right)$$

$$= -\frac{1}{4\pi}\int \vec{\nabla}'\bullet\left(\frac{\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right)d^3x' + \frac{1}{4\pi}\int \vec{\nabla}'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\bullet\vec{M}(\vec{x}')d^3x'$$

$$\left(\int \vec{\nabla}'\bullet\left(\frac{\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right)d^3x' = \oint \frac{\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}\bullet\hat{n}da' \to 0, S \to 0\right)$$

$$= -\frac{1}{4\pi}\int \vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\bullet\vec{M}(\vec{x}')d^3x'$$

$$\left(\because \vec{\nabla}\bullet(\vec{M}f) = f\left(\vec{\nabla}\bullet\vec{M}\right) + \vec{\nabla}f\bullet\vec{M}\right)$$

$$= -\frac{1}{4\pi}\vec{\nabla}\bullet\int \frac{\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x' \ldots\ldots(*)$$

§ 5.9 Methods of solving boundary-value problems in magnetostatics

Note that the expression (*) is generally applicable even for the limit of discontinuous contributions magnetic surface charge density:

$$\Phi_M(\vec{x}) = -\frac{1}{4\pi}\vec{\nabla}\bullet\int\frac{\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x'$$

$$= -\frac{1}{4\pi}\int\vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\bullet\vec{M}(\vec{x}')d^3x'$$

$$= \frac{1}{4\pi}\int\vec{\nabla}'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\bullet\vec{M}(\vec{x}')d^3x'$$

$$= -\frac{1}{4\pi}\int\frac{\vec{\nabla}'\bullet\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x' + \boxed{\frac{1}{4\pi}\int\vec{\nabla}'\bullet\left(\frac{\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right)d^3x'}$$

Surface charge contribution

§5.10 Uniformly magnetized sphere

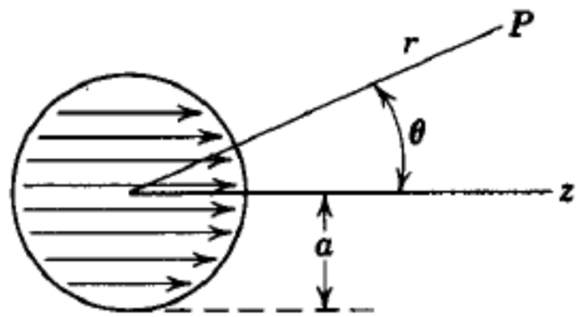
Consider a sphere of radius *a*, with a uniform permanent magnetization $\vec{M}$ of magnitude M_0 and parallel to the z-axis:

$$\vec{M} = M_0 \hat{z}$$

$$\because \vec{\nabla} \bullet \vec{M} = 0, \text{and } \sigma_M = \hat{n} \bullet \vec{M} = M_0 \cos\theta'$$

$$\therefore \Phi_M(\vec{x}) = \frac{1}{4\pi} \int \frac{M_0 \cos\theta'}{|\vec{x} - \vec{x}'|} a^2 d\Omega' = \frac{M_0 a^2}{4\pi} \int \frac{\cos\theta'}{|\vec{x} - \vec{x}'|} \sin\theta' d\theta' d\phi'$$

$$\text{remember}: \frac{1}{|\vec{x} - \vec{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma')$$



$$P_l(\cos\gamma') = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi)$$

$$\because \phi\text{-symmetry, and } \int_0^{2\pi} e^{im\phi} d\phi = 2\pi \delta_{m,0} \Rightarrow m = 0$$

§5.10 Uniformly magnetized sphere

$$\because Y_{l0}(\theta,\phi)=\sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)$$

$$\therefore \Phi_M(\vec{x})=\frac{M_0a^2}{4\pi}\sum_{l=0}^{\infty}\frac{r_<^{\,l}}{r_>^{\,l+1}}P_l(\cos\theta)\int_0^{2\pi}\int_0^{\pi}P_l(\cos\theta')\cos\theta'\sin\theta'\,d\theta'\,d\phi'$$

$$\text{By orthogonality}: \int_{-1}^{1}P_{l'}(x)P_l(x)dx=\frac{2}{2l+1}\delta_{l'l}\Rightarrow l=1$$

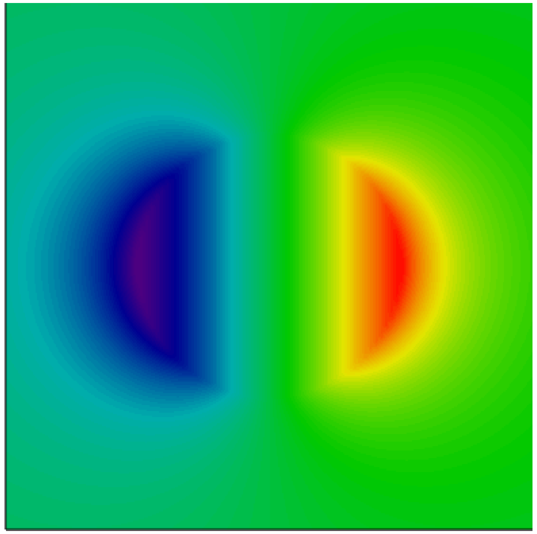
$$\therefore \Phi_M(\vec{x})=\frac{M_0a^2}{3}\frac{r_<^{\,l}}{r_>^{\,l+1}}\cos\theta$$

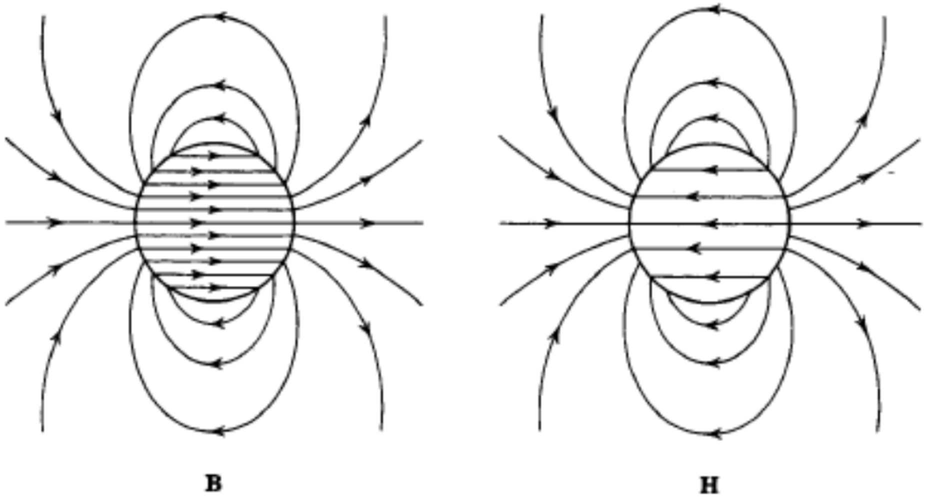
(1) Inside the sphere: $r_>=a,\ r_<=r\Rightarrow \Phi_M(r,\theta)=\frac{M_0}{3}r\cos\theta$

(2) Outside the sphere: $r_>=r,\ r_<=a\Rightarrow \Phi_M(r,\theta)=\frac{M_0a^3}{3r^2}\cos\theta$

§5.10 Uniformly magnetized sphere

The magnetic scalar potential and the lines of $\vec{B}$ and $\vec{H}$ are shown below. The lines of $\vec{B}$ are continuously closed paths, but those of $\vec{H}$ terminate on the surface because there is an effective surface charge density.





§5.11 Final remark

Find the magnetic dipole moment of a uniformly charged spherical shell of radius a rotating with angular frequency ω about the z-axis:

$$A = \pi(a\sin\theta)^2$$

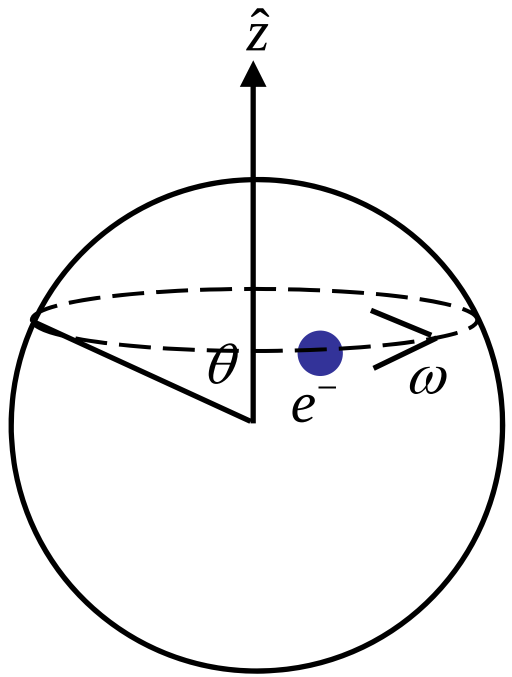
$$dI = \frac{\Delta q}{\Delta t} = \frac{\Delta q}{T} = \frac{2\pi}{T}\frac{\Delta q}{2\pi} = \frac{\omega}{2\pi}\sigma 2\pi a\sin\theta a d\theta$$

$$\therefore dm = \pi(a\sin\theta)^2 \cdot \frac{\omega}{2\pi}\sigma 2\pi a\sin\theta a d\theta$$

$$\therefore m = \pi\omega\sigma a^4 \int_0^{\pi} \sin^3\theta d\theta = \frac{4}{3}\pi\omega\sigma a^4$$

$$\therefore M = \frac{m}{V} = \omega\sigma a$$

$$\therefore \vec{M} = \omega\sigma a\hat{a}_z$$



We can use this model to explain some magnetized behavior of the atomic system.