III-C. Discrete-Time Fourier Transform

1 Review of DT Fourier Series of Periodic Sequences

A periodic DT signal $x[n]$ with period $N$ can be expanded as a sum of $N$ complex exponentials:

$$x[n] = \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N},$$

where the DT F. Series coefficient is

$$X_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}.$$

Note that

1. There are at most $N$ DT harmonics, while CT F. Series can have $\infty$ no. of harmonics.

2. $X_k = X_{k+N}$ is periodic, namely, $e^{j2\pi n}$ is indistinguishable from $e^{j(2\pi + 2\pi n)}$. In general, we only consider those complex exponentials with frequencies: $k\omega_0 \in [0, 2\pi)$ or $(-\pi, \pi]$.

3. (Oppenheim, Ex 3.12, p 218) For a periodic rectangular pulse train, $x[n] = 1$ for $-N_1 \leq n \leq N_1$, period = $N$.

2 DT Fourier Transform of Aperiodic Signals

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \iff x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

1. Proof: From DT Fourier series $X_k$ of a periodic sequence $x[n]$ to DT Fourier transform $X(e^{j\omega})$ of an aperiodic sequence $x[n]$ by letting the period $N \to \infty$.

$$x[n] = \sum_{k=1}^{N} X_k e^{j\omega_0 kn}, \quad \text{where } \omega_0 = \frac{2\pi}{N}$$

$$= \sum_{k=1}^{N} \left[ \frac{1}{N} \sum_{m}^{<N>} x[m] e^{-j\omega_0 km} \right] e^{j\omega_0 kn}$$

let $N \to \infty$, $\omega_0 N = 2\pi$, $\omega_0 \equiv d\omega \to 0$, and $k \omega_0 = \omega$

$$= \int_{\omega_0}^{2\pi} \frac{d\omega}{2\pi} \sum_{m=-\infty}^{\infty} x[m] e^{-j\omega m} e^{j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{\omega_0}^{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega \quad \text{(Inverse DTFT)}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = X(z)|_{z=e^{j\omega}}, \quad \text{(DTFT)}, \quad \text{where } X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$
2. $x[n]$ is a sampled sequence of a rectangular pulse train with width $2N_1 + 1$ and period $N$:

\[
\begin{array}{cccccccccccccccc}
\cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
\cdots & -N_1 & 0 & N_1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
\end{array}
\]

Let $2N_1 + 1 = 5$ and increase the period from $N = 10, 20, 40$, and so on, then the DT Fourier series coefficients (scaled by $N$) will approach the DTFT $X(\omega)$:

\[
NX_k = \frac{\sin(2\pi k(N_1 + 1/2)/N)}{\sin(\pi k/N)} \quad \text{when} \quad N \to \infty, \quad NX_k \to X(e^{j\omega}) = \frac{\sin(\omega(N_1 + \frac{1}{2}))}{\sin(\omega/2)}
\]

3. Examples:

(a) Ex 5.1. $a^n u[n] \iff \frac{1}{1 - ae^{-j\omega}}$, if $|a| < 1$.
(b) Ex 5.2. $a^n u[n] \iff 1 - \frac{1}{1 - ae^{-j\omega}} = \frac{1 - a^2}{2a \cos(\omega) + a^2}$, if $|a| < 1$.
(c) Ex 5.3. A rectangular pulse within $\pm N_1 \iff \sin(\omega(N_1 + \frac{1}{2})) \sin(\omega/2)$.
(d) Ex 5.4(a) $\delta[n] \iff 1$.
(e) Ex 5.4(b) Find the inverse DTFT of the rectangular pulse spectrum $F(e^{j\omega})$ bandlimited to $\pm W = \pm \pi/4$.

\[
f[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{j\omega}) e^{jn\omega} d\omega = \frac{1}{2\pi} \int_{-W}^{W} e^{jn\omega} d\omega = \frac{1}{j2\pi n} e^{jn\omega} \bigg|_{-W}^{W} = \frac{\sin Wn}{\pi n}.
\]

Compare with its duality: a rectangular sequence $\iff$ sinc-like spectrum (with aliasing) in Example 5.3(Figure 5.6).

3 DTFT of Periodic Sequences

1. Complex exponential: (See Figure 5.8)

\[
x[n] = e^{j\omega_0 n} \iff X(e^{j\omega}) = \sum_{k = -\infty}^{\infty} 2\pi \delta(\omega - \omega_0 - 2\pi k)
\]

(Pf): The inverse transform of $X(e^{j\omega})$ is

\[
x[n] = \frac{1}{2\pi} \int_{2\pi}^{2\pi} X(e^{j\omega}) e^{jn\omega} d\omega
\]

\[
= \frac{1}{2\pi} \sum_{k = -\infty}^{\infty} 2\pi \delta(\omega - \omega_0 - 2\pi k) e^{jn\omega} d\omega, \quad \text{only one impulse within the interval of } 2\pi
\]

\[
= e^{j(\omega_0 + 2\pi n)} = e^{j\omega_0 n}
\]

2. Any periodic sequence $x[n]$ with period $N$ can be represented with a Fourier series expansion as

\[
x[n] = \sum_{k = 0}^{N-1} X_k e^{jk(2\pi/N)n}
\]

Thus,

\[
X(e^{j\omega}) = DTFT\{x[n]\} = \sum_{k = 0}^{N-1} X_k \sum_{l = -\infty}^{\infty} 2\pi \delta(\omega - \frac{2\pi k}{N} - 2\pi l) = \sum_{k = -\infty}^{\infty} 2\pi X_k \delta(\omega - \frac{2\pi k}{N})
\]
3. Examples

(a) Ex 5.5 (See Figure 5.10)
\[
\cos \omega_0 n = \frac{1}{2} e^{j \omega_0 n} + \frac{1}{2} e^{-j \omega_0 n} \iff \pi \sum_{k=-\infty}^{\infty} [\delta(\omega - \omega_0 - 2\pi k) + \delta(\omega + \omega_0 - 2\pi k)]
\]

(b) Ex 5.6 (See Figure 5.11)
\[
\sum_{k=-\infty}^{\infty} \delta[n - kN] \iff 2\pi N \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi kN}{N})
\]

4 DTFT Properties

1. DTFT \(X(e^{j\omega})\) is a continuous function of the frequency \(\omega\).
2. DTFT \(X(e^{j\omega})\) is a periodic function of \(\omega\) with period \(2\pi\).
   
   Because \(X(\omega) = X(\omega + 2\pi) = X(\omega + 4\pi) = \ldots = X(\omega - 2\pi) = \ldots\)
3. DTFT is linear: \(ax[n] + by[n] \iff aX(e^{j\omega}) + bY(e^{j\omega})\).
4. Time shift: \(x[n - n_0] \iff e^{-j\omega n_0} X(e^{j\omega})\).
5. Frequency shift: \(e^{j\omega_0 n} x[n] \iff X(e^{j(\omega - \omega_0)})\).
   
   [Ex5.7] Convert a LowPass filter to a HighPass filter:
   
   \[
   H_{hpf}(\omega) = H_{lpf}(\omega - \pi)
   \]
   
   \[
   h_{hpf}[n] = e^{j\pi n} h_{lpf}[n] = (-1)^n h_{lpf}[n]
   \]
   
   which means that if \(h_{lpf}[n] = 1, 1, 1, 1, 1, 1, 0000\) is a LPF, then we can easily obtain a HPF \(h_{hpf}[n] = 1, -1, 1, -1, 1, -1, 0000\) by changing the odd-numbered coefficients of the LPF. (See Figure 5.12)
6. Conjugation: \(x^*[n] \iff X^*(e^{-j\omega})\)
7. If \(x[n]\) is a real-valued sequence, then
   
   (a) \(X(e^{j\omega})\) is conjugate-symmetric, \(X(e^{j\omega}) = X^*(e^{-j\omega})\),
   
   (b) amplitude spectrum is even: \(|X(e^{j\omega})| = |X(e^{-j\omega})|\),
   
   (c) phase spectrum is odd: \(\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})\).
8. Time reversal: \(x[-n] \iff X(e^{-j\omega})\).
9. Differentiation in Frequency: \(nx[n] \iff j \frac{dX(e^{j\omega})}{d\omega}\)
10. Parseval’s relation:
\[
\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\omega})|^2 d\omega
\]
11. Convolution property:
\[
\text{If } y[n] = x[n] * h[n] \text{ then } Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}).
\]
12. Multiplication: If \(z[n] = x[n]y[n]\), then \(Z(\omega) = \frac{1}{2\pi} \int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta\).
13. Time-scaling? \(x[\frac{n}{2}] \iff X(e^{jk\omega})\). What is \(x[kn] \iff ?\)
5 DT LTI systems: difference equations

For a DT LTI system satisfying an $N$-th order difference equation:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Its frequency response is given by

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{jk\omega}}{\sum_{k=0}^{N} a_k e^{jk\omega}}$$

[Pt]

1. Denote $h[n]$ as the impulse response, then $y[n] = x[n] * h[n]$.
2. By the convolution property, $Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega})$, where $H(e^{j\omega})$ is the DTFT of $h[n]$.
3. Apply DTFT to both sides and from the time-delay property($y[n-k]$ replaced by $e^{-jk\omega}Y(e^{j\omega})$):

$$\sum_{k=0}^{N} a_k e^{-jk\omega}Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega}X(e^{j\omega})$$

4. Now the frequency response is simply the ratio of output’s and input’s spectra:

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{jk\omega}}{\sum_{k=0}^{N} a_k e^{jk\omega}}$$

[Ex 5.19] For a DT LTI system: $y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$.

1. Take DTFT, we have

$$Y(e^{j\omega}) - \frac{3}{4}e^{-j\omega}Y(e^{j\omega}) + \frac{1}{8}e^{-j2\omega}Y(e^{j\omega}) = 2X(e^{j\omega})$$

2. Its frequency response is $H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}}$.

3. By partial fractional expansion,

$$H(e^{j\omega}) = \cdots = \frac{4}{1 - \frac{1}{2}e^{-j\omega}} - \frac{2}{1 - \frac{1}{4}e^{-j\omega}}$$

4. From Inverse DTFT, its impulse response is $h[n] = \text{DTFT}^{-1}\{H(e^{j\omega})\} = 4(\frac{1}{2})^nu[n] - 2(\frac{1}{4})^nu[n]$.

Take a look at the difference equation again. It appears to be difficult in solving the impulse response $h[n]$ by directly plugging $x[n] = \delta[n]$. Thanks to the tool of F.T., we can solve it in the transform frequency domain. $Z$ transform is another useful technique.