3.2 The Method of Images

Please refer to D. K. Cheng’s “Field and Wave Electromagnetics”, Chapt. 4-4. It’s a more complete section for the introduction to the method of images.

3.2.1 The Classic Image Problem

Suppose a point charge \( q \) is held a distance \( d \) above an infinite grounded conducting plane. What is the potential in the region above the plane?

In the \( z > 0 \) region, we plan to solve the Poisson’s equation with a single point charge at \((0, 0, d)\).

The boundary condition is:
1. \( V = 0 \) when \( z = 0 \)
2. \( V \to 0 \) when \( \sqrt{x^2 + y^2 + z^2} \to \infty \)

Guessed solution: we firstly obtain the electric potential of two charges, one has \( +q \) charge at \((0,0,d)\) and the other has \( -q \) at \((0,0,-d)\).

The guessed potential is:

\[
V(x, y, z) = \frac{1}{4\pi \varepsilon_0} \left( \frac{+q}{\sqrt{x^2 + y^2 + (z-d)^2}} + \frac{-q}{\sqrt{x^2 + y^2 + (z+d)^2}} \right).
\]

We find that the potential is \( V = 0 \) when \( z = 0 \) or \( \sqrt{x^2 + y^2 + z^2} \to \infty \).

With the help of uniqueness theorem, we are sure that we got the right answer.

3.2.2 Induced Surface Charge

\[
\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \rightarrow (E_{\text{above}} - E_{\text{below}}) \Lambda = \frac{Q_{\text{enc}}}{\varepsilon_0} = \frac{A \sigma}{\varepsilon_0}
\]

\[E_{\text{above}} = \frac{\sigma}{\varepsilon_0} \]

We know \( V_{\text{above}}(x, y, z) = \frac{1}{4\pi \varepsilon_0} \left( \frac{+q}{\sqrt{x^2 + y^2 + (z-d)^2}} + \frac{-q}{\sqrt{x^2 + y^2 + (z+d)^2}} \right) \).

\[
\sigma = \varepsilon_0 E_{\text{above}} = -\varepsilon_0 \left. \frac{\partial V_{\text{above}}}{\partial z} \right|_{z=0} = -\frac{1}{4\pi} \left( \frac{2qd}{(x^2 + y^2 + d^2)^{3/2}} \right).
\]
The induced total charge: \( Q = \int_{0}^{\frac{\phi}{d}} -\frac{2qdr}{4\pi \left(r^2 + d^2\right)^{\frac{3}{2}}} + \int_{0}^{\frac{\phi}{d}} -\frac{2qdr}{4\pi \left(r^2 + d^2\right)^{\frac{3}{2}}} \)

3.2.3 Force and Energy

**Force:** \( \vec{F}_{\text{on-q}} = \frac{1}{4\pi \varepsilon_0} \left( +q \right) \left(-q\right) \hat{\mathbf{z}}_2 \)

The force required to move the charge is

\[ \vec{F} = \frac{1}{4\pi \varepsilon_0} \frac{q^2}{4d^2} \hat{\mathbf{z}} \]

**Energy:**

\[ W = \int \vec{F} \cdot d\vec{l} = \int_{0}^{d} \frac{1}{4\pi \varepsilon_0} \frac{q^2}{(2z)^2} dz = \frac{1}{4\pi \varepsilon_0} \frac{q^2}{4d} \]

The energy stored is not \( W = -\frac{1}{4\pi \varepsilon_0} \frac{q^2}{2d} \) but half of it. Since the \(-q\) charge is an induced charge, not a real charge. You can also look at the problem by \( W = \frac{\varepsilon_0}{2} \int E^2 d\tau \). The space that have non-zero electric field is in the region of \( z > 0 \), so the work needed is only one half of energy required for two-point-charge configuration.

Example: Two semi-infinite grounded conducting planes meet at right angles. In the region between them, there is a point charge \( q \). What is the potential? What is the energy taken to arrange the point charge?

\[ W = \frac{1}{2 \frac{1}{4\pi \varepsilon_0}} \left( -\frac{q^2}{(2a)^2} + -\frac{q^2}{(2b)^2} + -\frac{q^2}{\sqrt{(2a)^2 + (2b)^2}} \right) \]

Symmetry consideration:

\[ W = \frac{1}{4 \frac{1}{4\pi \varepsilon_0}} \left( -\frac{q^2}{(2a)^2} + -\frac{q^2}{(2b)^2} + -\frac{q^2}{\sqrt{(2a)^2 + (2b)^2}} + -\frac{q^2}{(2a)^2} + -\frac{q^2}{(2b)^2} + -\frac{q^2}{\sqrt{(2a)^2 + (2b)^2}} \right) \]

3.2.4 Other Image Problems

A Conducting Sphere with a Point Charge
We find that someone have noticed that the field of two unequal point charges has an equipotential that is a sphere.

Assume the center of sphere is at the origin and the image charge is placed at a distance of \( b \).

The 1\(^{\text{st}}\) Point of View:

Zero potential at the surface \( r = \frac{q + q'}{r} = 0 \Rightarrow r' = \frac{-q'}{q} = \text{const} \)

If \( \frac{r'}{r} = \text{const} \), then \( \sqrt{\frac{R^2 + b^2 - 2Rb \cos \theta}{R^2 + a^2 - 2Ra \cos \theta}} = \text{const} \) and independent on \( \theta \).

The only solutions is \( \sqrt{\frac{b}{a} \left( \frac{R^2}{b} + ab - 2Ra \cos \theta \right)} = \text{const} \).

\( b = \frac{R^2}{a} \), \( q' = -\frac{r'}{r} = -\sqrt{\frac{b}{a} = -\frac{R}{a}} \Rightarrow q' = -\frac{R}{a} q \) and \( b = \frac{R^2}{a} \)

The 2\(^{\text{nd}}\) Point of View:

The image charge \( q' \) should have the same magnitude as \( q \) if the radius goes to infinity.

\( a \rightarrow R \rightarrow \infty \), \( q' = -q \) and \( R \rightarrow 0 \), \( q' \rightarrow 0 \) \( q' = -Rq \frac{1}{a} \)

\[
V(\vec{r}) = \frac{1}{4\pi \varepsilon_0} \left( \frac{q}{\Pi} + \frac{q'}{\Pi'} \right)
\]

\[
F = \frac{1}{4\pi \varepsilon_0} \frac{q' q}{(a - b)^2} = \frac{1}{4\pi \varepsilon_0} \left( -\frac{R^2}{a} q \ast q \right)
\]

If the conducting sphere is not grounded, we can put some charge at the origin to solve the problem.

Superposition and the concept of symmetry:
Charged Sphere in front of the Grounded Plane

\[ \begin{align*}
+Q &\rightarrow \text{Grounded Plane} \rightarrow -Q \\
-Q &\rightarrow \text{Conducting Sphere} \rightarrow \frac{R}{a}Q \quad \text{at} \quad \frac{R^2}{a}
\end{align*} \]

\[ -\frac{R}{a}Q \rightarrow \text{Grounded Plane} \rightarrow -\frac{R}{a}Q \quad \text{at} \quad a - \frac{R^2}{a} \]

\[ -\frac{R}{a}Q \rightarrow \text{Conducting Sphere} \rightarrow \frac{R}{a}Q \left( \frac{R}{a - \frac{R^2}{a}} \right) \quad \text{at} \quad \frac{R^2}{a - \frac{R^2}{a}} \]

Line Charge and Parallel Conducting Cylinder

\[ V = -\int_{a}^{r} \frac{\lambda}{2\pi\varepsilon_0} \, dr = \frac{\lambda}{2\pi\varepsilon_0} \ln \left( \frac{a}{s} \right) \]

Find the cylindrical symmetry of electric potential:

\[ V = \frac{\lambda}{2\pi\varepsilon_0} \ln \left( \frac{a}{r} \right) + \frac{\lambda_{ind}}{2\pi\varepsilon_0} \ln \left( \frac{a}{r'} \right) \]

We can find cylindrical symmetry if \( \lambda_{ind} = -\lambda \).

\[ V = \frac{\lambda}{2\pi\varepsilon_0} \ln \left( \frac{r'}{r} \right), \quad \frac{r'}{r} = \sqrt{\frac{R^2 + b^2 - 2Rb\cos\theta}{R^2 + a^2 - 2Ra\cos\theta}} \]

It is the same as the spherically symmetrical problem.

\[ b = \frac{R^2}{a} \quad \text{but} \quad \lambda_{ind} = -\lambda \]

Example: Determine the capacitance per unit length between two long parallel, circular conducting wires of radius \( a \). The axes of the wires are separated by a distance \( D \).
The only solution is that one carries $\lambda$ while the other carries $-\lambda$ line charge.

$$\begin{align*}
V_L &= \frac{\lambda}{2\pi \varepsilon_0} \ln \left( \frac{r}{r'} \right) = \frac{\lambda}{2\pi \varepsilon_0} \ln \left( \frac{a}{d} \right), \quad V_R = -\frac{\lambda}{2\pi \varepsilon_0} \ln \left( \frac{a}{d} \right), \quad V = V_R - V_L \\
C &= \frac{\lambda}{\pi \varepsilon_0} \ln \left( \frac{d}{a} \right), \quad d = D - b = D - \frac{a^2}{d} \quad \Rightarrow \quad d = \frac{1}{2} \left( D + \sqrt{D^2 - 4a^2} \right)
\end{align*}$$

**Exercise:** 7, 10, 11