1.2 Differential Calculus

1.2.1 “Ordinary” Derivatives

\[ f = f(x), \quad df = \left( \frac{df}{dx} \right) dx \rightarrow \Delta y = \frac{dy}{dx} \Delta x \]

1.2.2 Gradient

We know that some place is higher (more concentrated) and we try to find the scalar function (height or concentration) as a function of x and y coordinates. After we get the scalar function, we can calculate how they are formed by gradient. The gradient means flowing from concentrated places to dilute places. The gradient means some forces that build the final geometry.

The gradient \( f \) is a vector quantity.

The gradient points in the direction of maximum increase of the scalar function.

The magnitude \( |\nabla f| \) gives the slope (rate of increase) along this maximal direction.

```math
\nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} \quad & \quad d\vec{l} = \hat{i} dx + \hat{j} dy + \hat{k} dz \rightarrow df = \nabla f \cdot d\vec{l} = |\nabla f| |d\vec{l}| \cos \theta
\```
Example: Find the gradient of $r = r(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.

Problem 1.14

1.2.3 The Operator $\nabla$

Give you the concept of operator that may be used in quantum mechanics.

\[ \nabla T = \left( \nabla \right) ^T , \text{ where } \nabla \text{ is a vector and } T \text{ is a scalar function of } (x, y, z, \ldots). \]

\[ \nabla = \left( \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right), \text{ which is like that } \frac{df}{dx}, \text{ we may say } \frac{d}{dx} \text{ is an operator acting on } f. \]

The $\nabla$ operator is like an ordinary vector and having the following properties:

1. on a scalar function $f$: $\nabla f$ (the gradient)
2. on a vector function $\vec{v}$ – $\vec{A} \cdot \vec{B} = c$, via the dot product: $\nabla \cdot \vec{v}$ (the divergence)
3. on a vector function $\vec{v}$ – $\vec{A} \times \vec{B} = \vec{C}$, via the cross product: $\nabla \times \vec{v}$ (the curl)

The operator is not really an ordinary vector, for example, $\nabla \cdot (c \vec{v}) \neq c \nabla \cdot \vec{v}$ if $c$ is a scalar function.

1.2.4 The Divergence

Why you have vector forces to drive objects out? There must be something inside.

Uniform field:

Non-uniform field:

The divergence expression is

\[ \nabla \cdot \vec{v} = \left( \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \cdot \left( \hat{x} v_x + \hat{y} v_y + \hat{z} v_z \right) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \]

Example 1.4:

$\vec{v}_a = \vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$, $\vec{v}_b = \hat{z}$, $\vec{v}_c = z\hat{z}$. Calculate their divergence.

Why does $\nabla \cdot \vec{v}_a = 3$ exist everywhere? It is like the electric field inside the uniformly distributed charge sphere.

Problem 1.17
1.2.5 The Curl

Why the vector rotates in a field? There must be something inside. How can we measure the rotation of a vector function?

\[
\frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y}
\]

\[
\nabla \times \vec{v} = \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
v_x & v_y & v_z
\end{vmatrix}
\]

Example:
\[
\vec{v}_a = -y\hat{x} + x\hat{y}, \quad \vec{v}_b = x\hat{y}.
\]
Calculate their curls.

1.2.6 Product Rules

They are something like chain rules.

Use the concept that take the \(\frac{d}{dx}\) as an operator.

The sum rule: \(\frac{d}{dx}(f + g) = \frac{df}{dx} + \frac{dg}{dx}\)

The rule for multiplying by a constant: \(\frac{d}{dx}(cf) = c\frac{df}{dx}\)

The product rule: \(\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}\)

The quotient rule: \(\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}\)

There are similar relations for the gradient, the divergence, and the curl.

The sum rule:
\nabla(f + g) = \nabla f + \nabla g, \quad \nabla \cdot (\vec{A} + \vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B}, \quad \nabla \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B}
\n
The rule for multiplying by a constant:
\nabla(cf) = c\nabla f, \quad \nabla \cdot (c\vec{A}) = c\nabla \cdot \vec{A}, \quad \nabla \times (c\vec{A}) = c\nabla \times \vec{A}
\nSix product rules:
1. \(\nabla(fg) = g\nabla f + f\nabla g\) - the operator always apply on the right term
2. \( \nabla(\vec{A} \cdot \vec{B}) = ? \) The answer is a vector -> what combination of \( \nabla, \vec{A}, \vec{B} \) could be a vector? There is no simple product of \( \nabla \cdot \vec{A} \) in the answer, and again, the operator acts on the right term.
\[
\nabla(\vec{A} \cdot \vec{B}) = \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A}) + (\vec{A} \cdot \nabla)\vec{B} + (\vec{B} \cdot \nabla)\vec{A}
\]

3. \( \nabla \cdot (f\vec{A}) = \vec{A} \cdot (\nabla f) + f(\nabla \cdot \vec{A}) \)

4. \( \nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}) \)

5. \( \nabla \times (f\vec{A}) = f(\nabla \times \vec{A}) - \vec{A} \times (\nabla f) \)

6. \( \nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B} \)

Three quotient rules:
\[
\nabla \left( \frac{f}{g} \right) = \frac{g(\nabla f) - f(\nabla g)}{g^2} \quad \text{(like the 1st product rule)}
\]
\[
\nabla \cdot \left( \frac{\vec{v}}{g} \right) = \frac{g(\nabla \cdot \vec{v}) - \vec{v} \cdot (\nabla g)}{g^2} \quad \text{(like the 3rd product rule)}
\]
\[
\nabla \times \left( \frac{\vec{v}}{g} \right) = \frac{g(\nabla \times \vec{v}) + \vec{v} \times (\nabla g)}{g^2}
\]

1.2.7 Second Derivatives

1. The Laplacian: \( \nabla \cdot (\nabla f) = \nabla^2 f = \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f + \frac{\partial^2}{\partial z^2} f \), it can also apply on a vector function
\[
\frac{\hat{x}}{\frac{\partial}{\partial x}} \quad \frac{\hat{y}}{\frac{\partial}{\partial y}} \quad \frac{\hat{z}}{\frac{\partial}{\partial z}}
\]

2. \( \nabla \times (\nabla f) = 0 \) (since \( \vec{A} \times \vec{A} = 0 \) ??),
\[
\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \quad \frac{\partial f}{\partial z}
\]

3. \( \nabla(\nabla \cdot \vec{v}) \neq (\nabla \cdot \nabla)\vec{v} \)

4. \( \nabla \cdot (\nabla \times \vec{v}) = 0 \)

5. \( \nabla \times (\nabla \times \vec{v}) = ? \) Use \( \nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B} \), take
\[
\vec{A} = \nabla, \quad \text{and remove zero terms:} \quad \vec{B}(\nabla \cdot \vec{v}) \& (\nabla \cdot \vec{v})\vec{v} \rightarrow
\]
\[ \nabla \times (\nabla \times \vec{v}) = \nabla (\nabla \cdot \vec{v}) - \nabla^2 \vec{v} \]

Just write down it in detail, so you can prove it.

\[
\nabla \cdot (\nabla \times \vec{v}) = \nabla \cdot \left( \hat{x} \left( \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) + \hat{y} \left( \frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z} \right) + \hat{z} \left( \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \right)
\]

\[
= \frac{\partial^2 v_z}{\partial x \partial y} - \frac{\partial^2 v_y}{\partial x \partial z} + \frac{\partial^2 v_z}{\partial y \partial z} - \frac{\partial^2 v_x}{\partial y \partial z} + \frac{\partial^2 v_y}{\partial z \partial x} - \frac{\partial^2 v_z}{\partial z \partial y} v_x = 0
\]