17. Discrete Fourier Transform

In previous topics, both the Fourier transform and the discrete-time Fourier transform produce continuous functions of frequency. In order to use the digital computer, we need a different transformation which can generate discrete-frequency spectrum for discrete-time samples. One of the transformation is called the discrete Fourier transform or DFT in brief.

It has been introduced that a continuous-time function \( x(t) \) can be represented by its sampled values \( x[k] = x(kT) \) where \( T \) is the sampling time. The pair of DTFT and IDTFT are expressed as

\[
X(\Omega) = \sum_{k=-\infty}^{\infty} x[k] e^{-j\Omega k} \tag{1}
\]

\[
x[k] = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} X(\Omega) e^{j\Omega k} d\Omega \tag{2}
\]

where \( X(\Omega) \) is a periodic function with period \( 2\pi \). To use the computer, we have to apply the approximate signal of \( x[k] \), expressed as

\[
x_N[k] = \begin{cases} x[k] & 0 \leq k \leq N - 1 \\ 0 & \text{elsewhere} \end{cases} \tag{3}
\]

which contains \( N \) values of \( x[k] \). Besides, the continuous \( X(\Omega) \) should be modified by discrete-frequency samples shown as

\[
X_N(\Omega) = \sum_{k=-\infty}^{\infty} x_N[k] e^{-j\Omega k} = \sum_{k=0}^{N-1} x[k] e^{-j\Omega k} \tag{4}
\]

which is approximated to \( X(\Omega) \) if not all the values \( x[k] \) are included in \( x_N[k] \). Since \( X_N(\Omega) \) is still a periodic function with period \( 2\pi \), we generally select \( N \) samples of \( X_N(\Omega) \), the same number of samples of \( x_N[k] \), to represent the frequency spectrum in one period \( 2\pi \). Therefore, the sampled frequency spectrum is given by

\[
X_s(\Omega) = X_N(\Omega) \sum_{n=0}^{N-1} \delta\left( \Omega - 2\pi \frac{n}{N} \right) \tag{5}
\]

which results in

\[
X_s(\Omega) = \sum_{n=0}^{N-1} X_N\left( 2\pi \frac{n}{N} \right) \delta\left( \Omega - 2\pi \frac{n}{N} \right) \tag{6}
\]

For convenience, we let \( X[n] = X_N\left( 2\pi \frac{n}{N} \right) \), for \( n=0,1,2,\ldots,N-1 \), and from (4) the
discrete Fourier transform is defined as

\[ \mathfrak{D}_D \{ x[k] \} = X[n] = X_N(2\pi \frac{n}{N}) = \sum_{k=0}^{N-1} x[k] e^{-j\frac{2\pi kn}{N}} \]  

(7)

Now, calculate the following summation

\[ \sum_{n=0}^{N-1} X[n] e^{\frac{j2\pi n}{N}} = \sum_{n=0}^{N-1} \left( \sum_{m=0}^{N-1} x[m] e^{\frac{-j2\pi m}{N}} \right) e^{\frac{j2\pi kn}{N}} \]  

\[ = \sum_{m=0}^{N-1} x[m] \sum_{n=0}^{N-1} e^{-j(m-k)n} \frac{2\pi}{N} \]  

(8)

Since

\[ \sum_{n=0}^{N-1} e^{-j(m-k)n} \frac{2\pi}{N} = \begin{cases} N & m = k \\ 0 & m \neq k \end{cases} \]  

(9)

we rewrite (8) as

\[ \sum_{n=0}^{N-1} X[n] e^{\frac{j2\pi n}{N}} = N x[k] \]  

(10)

Therefore, we have the inverse discrete Fourier Transform, or IDFT in brief, as below:

\[ \mathfrak{D}_D^{-1} \{ X[n] \} = x[k] = \frac{1}{N} \sum_{n=0}^{N-1} X[n] e^{\frac{j2\pi kn}{N}} \]  

(11)

To sum up, the transform pair developed for \( N \) discrete-frequency sampled calculated from \( N \) discrete-time samples is known as \( DFT \) and \( IDFT \), both respectively expressed by

\[ \mathfrak{D}_D \{ x[k] \} = X[n] = \sum_{k=0}^{N-1} x[k] e^{-j\frac{2\pi kn}{N}}, n=1, 2, \ldots, N-1; \]  

(11)

\[ \mathfrak{D}_D^{-1} \{ X[n] \} = x[k] = \frac{1}{N} \sum_{n=0}^{N-1} X[n] e^{\frac{j2\pi kn}{N}}, k=1, 2, \ldots, N-1. \]  

(12)

Moreover, we often define \( W_N = e^{-\frac{2\pi}{N}} \) and thus

\[ \sum_{n=0}^{N-1} W_N^{kn} = \begin{cases} N & k = 0 \\ 0 & k = 1, 2, \ldots, N-1 \end{cases} \]  

(13)

In addition, (11) and (12) can be rewritten into

\[ \mathfrak{D}_D \{ x[k] \} = X[n] = \sum_{k=0}^{N-1} x[k] W_N^{kn}, n=1, 2, \ldots, N-1; \]  

(14)

\[ \mathfrak{D}_D^{-1} \{ X[n] \} = x[k] = \frac{1}{N} \sum_{n=0}^{N-1} X[n] W_N^{-kn}, k=1, 2, \ldots, N-1. \]  

(15)
In general, both the DFT and IDFT are respectively stated by (14) and (15).

Since the DFT $X[n]$ has a resolution of $2\pi/N$ for $N$ samples of $x[k]$, if a DFT of higher resolution is required, we could use the zero padding process to increase the number $N$ by adding zero-valued samples to $x[k]$.

**Example** Find the DFT for $x[k]=k+1$, $k=0,1,2,3$, for $N=4$ and $N=8$.

**Sol:**

(A) $N=4$, $W_4 = e^{-j2\pi/4} = -j$, $X[n] = \sum_{k=0}^{3} x[k] W_4^{nk}$

\[
\]

\[
X[1] = x[0] + x[1](-j) + x[2](-j)^2 + x[3](-j)^3 = 1 - 2j - 3 + 4j = -2 + 2j
\]

\[
X[2] = x[0] + x[1](-j)^2 + x[2](-j)^4 + x[3](-j)^6 = 1 - 2 + 3 - 4 = -2
\]

\[
X[3] = x[0] + x[1](-j)^3 + x[2](-j)^6 + x[3](-j)^9 = 1 + 2j - 3 - 4j = -2 - 2j
\]

This MATLAB program is for DFT of N=4
\[
>> N=4;
>> x=[1 2 3 4];
>> for kk=1:N
    X(kk)=0;
    k=kk-1;
    for nn=1:N;
        n=nn-1;
        X(kk)=X(kk)+x(nn)*exp(-j*2*pi*k*n/N);
    end
    end
\]
\[
>> x
x =
1 2 3 4
\]
\[
>> X
X =
10.0000 -2.0000 + 2.0000i -2.0000 - 0.0000i -2.0000 - 2.0000i
\]

(A) $N=8$, $W_8 = e^{-j2\pi/8} = e^{-j\pi/4}$, $X[n] = \sum_{k=0}^{7} x[k] W_8^{nk}$

\[
\]

\[
    = 1 + \sqrt{2}(1-j) + 3(-j) + 2\sqrt{2}(-1-j) = 1 - \sqrt{2} - j(3 + 3\sqrt{2})
\]

\[
    = 1 + 2(-j) + 3(-1) + 4(j) = -2 + j2
\]

\[
    = 1 + \sqrt{2}(-1-j) + 3(j) + 2\sqrt{2}(1-j) = 1 + \sqrt{2} + j(3 - 3\sqrt{2})
\]

\[
X[4] = ........
\]
This MATLAB program is for DFT of N=8
>> N=8;
>> x=[1 2 3 4 0 0 0 0];
>> for kk=1:N
    X(kk)=0;
    k=kk-1;
    for nn=1:N;
        n=nn-1;
        X(kk)=X(kk)+x(nn)*exp(-j*2*pi*k*n/N);
    end
end
>> x
x =
   1   2   3   4   0   0   0   0
>> X
X =
Columns 1 through 4
   10.0000  -0.4142 - 7.2426i  -2.0000 + 2.0000i  2.4142 - 1.2426i
Columns 5 through 8
   -2.0000 - 0.0000i  2.4142 + 1.2426i  -2.0000 - 2.0000i  -0.4142 + 7.2426i

The DFT of \( N \)-sample \( x[k] \) requires \( N^2 \) multiplication. To compute DFT more efficiently, we often employ the so-called fast Fourier transform, FFT in brief, which compute DFT with \( N=2^m \) samples and approximately requires \( N (\log_2 N) \) multiplication, much less than \( N^2 \). There are three important properties of \( W_N = e^{-j \frac{2\pi}{N}} \) for \( N=2^m \), which are

\[
W_N^N = e^{-j \frac{2\pi}{N} N} = e^{-j 2\pi} = 1 \quad (16)
\]

\[
W_N^{N/2} = e^{-j \frac{2\pi}{N} \frac{N}{2}} = e^{-j \pi} = -1 \quad (17)
\]

\[
W_N^{(N/2)+m} = -W_N^m \quad (18)
\]

Based on the above properties, below show the process of FFT for \( N=2, 4 \) and 8.

\( N=2, \quad W_2 = e^{-j \frac{2\pi}{2}} = -1 \)

\[
\begin{align*}
X[0] &= x[0]W_2^0 + x[1]W_2^1 = x[0] + x[1] \\
\end{align*}
\]
$N = 4$, $W_4 = e^{-2\pi i/4} = -j$

\[
\begin{align*}
\end{align*}
\]

$X[0] = (x[0] + x[2]) + (x[1] + x[3]) = x_{0+2} + x_{1+3}$

$X[1] = (x[0] - x[2]) + (x[1] - x[3])W_4^1 = x_{0-2} + x_{1-3}W_4$

$X[2] = (x[0] + x[2]) - (x[1] + x[3]) = x_{0+2} - x_{1+3}$

$X[3] = (x[0] - x[2]) - (x[1] - x[3])W_4^1 = x_{0-2} - x_{1-3}W_4$

$N = 8$, $W_8 = e^{-2\pi i/8} = -j$

\[
\begin{align*}
\end{align*}
\]


\[
X[0] = (x[0] + x[4]) + (x[1] + x[5]) + (x[2] + x[6]) + (x[3] + x[7])
\]
\[
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