16. Discrete-Time Fourier Transform

In sampling process, the continuous signal \( x(t) \) is transformed to the discrete-time signal \( x_s(t) \) by the use of periodic impulse train \( \delta(t) \) with period \( T \) and frequency \( \omega_0 = \frac{2\pi}{T} \). The resulted discrete-time signal is expressed as

\[
x_s(t) = x(t)\sum_{k=-\infty}^{\infty} \delta(t-kT) = \sum_{k=-\infty}^{\infty} x(t)\delta(t-kT)
\]

(1)

and taking Fourier transform yields

\[
X_s(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega-k\omega_0)
\]

(2)

It is known that the recovery of band-limited \( x(t) \) from \( x_s(t) \) can be explained by Shannon’s sampling theorem. In addition to sampling process, the discrete-time signal \( x_s(t) \) can be also used to define the discrete-time Fourier transform which will be discussed in this topic.

From (1) and the truth of \( x(t)\delta(t-kT) = x(kT)\delta(t-kT) \), the sampled signal can be rewritten as

\[
x_s(t) = \sum_{k=-\infty}^{\infty} x(kT)\delta(t-kT)
\]

(3)

whose Fourier transform is expressed as

\[
X_s(\omega) = \sum_{k=-\infty}^{\infty} x(kT)e^{-jk\omega T}
\]

(4)

Let \( x[k] = x(kT) \) and \( \omega T = \Omega \) and then (4) becomes

\[
X(\Omega) = \mathcal{F}\left\{ X_s(\frac{\Omega}{T}) \right\} = \sum_{k=-\infty}^{\infty} x[k]e^{-jk\Omega}
\]

(5)

where \( X(\Omega) \) is the so-called discrete-time Fourier transform or DTFT in short. Here, we also express it as

\[
\mathcal{F}_D \{ x[k] \} = X(\Omega) = \sum_{k=-\infty}^{\infty} x[k]e^{-jk\Omega}
\]

(6)

Further, take the following integration

\[
\int_{\Omega_1}^{\Omega_1+2\pi} X(\Omega)e^{jm\Omega}d\Omega = \int_{\Omega_1}^{\Omega_1+2\pi} \sum_{k=-\infty}^{\infty} x[k]e^{-jk\Omega}e^{jm\Omega}d\Omega
\]

(7)

\[
= \sum_{k=-\infty}^{\infty} x[k]\int_{\Omega_1}^{\Omega_1+2\pi} e^{j(m-k)\Omega}d\Omega
\]
which leads to
\[
\int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega = 2\pi \cdot x[n]
\]  
(8)

since
\[
\int_{-\pi}^{\pi} e^{j(n-k)\Omega} d\Omega = \begin{cases} 2\pi & k = n \\ 0 & k \neq n \end{cases}
\]  
(9)

From (8), it is obvious that
\[
\mathcal{Z}_{DT}^{-1}\{X(\Omega)\} = x[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega k} d\Omega
\]  
(10)

which is the inverse discrete-time Fourier transform, or IDTFT in short.

Now, let’s compare (6) to the z-transform \( \mathcal{Z}\{x[k]\} = \sum_{k=0}^{\infty} x[k] \cdot z^{-k} \), we can find that
\[
\mathcal{Z}_{DT}\{x[k]\} = X(\Omega) = \sum_{k=0}^{\infty} x[k] e^{-j\Omega k} = \sum_{k=0}^{\infty} x[k] \cdot z^{-k} \bigg| \quad z = e^{j\Omega}
\]  
(11)

In other words, we can use all the properties of z-transform satisfying \( z = e^{j\Omega} \) for the DTFT of \( x[k] \) for \( k \geq 0 \).

For example, the z-transform of \( a^k \) for \( k \geq 0 \) is \( \frac{z}{z-a} \) where \( |z| > |a| \). By the use of \( z = e^{j\Omega} \), we can obtain
\[
\mathcal{Z}_{DT}\{a^k\} = \left. \frac{z}{z-a} \right|_{z = e^{j\Omega}} = \frac{e^{j\Omega}}{e^{j\Omega} - a} = \frac{1}{1 - ae^{-j\Omega}} , \quad |a| < 1
\]  
(12)

Note that (12) is only true for \( |a| < 1 \) since \( |z| = |e^{j\Omega}| = 1 \). That implies for \( k \geq 0 \) we have \( \mathcal{Z}_{DT}\{0.5^k\} = \frac{1}{1 - 0.5e^{-j\Omega}} \) and \( \mathcal{Z}_{DT}\{2^k\} \) does not exist.

Some important properties related to the discrete-time Fourier transform will be further discussed below:

1) Periodicity
\[
X(\Omega + 2\pi) = X(\Omega)
\]  
(13)

2) Linearity
\[
\mathcal{Z}_{DT}\{ax[k] + by[k]\} = a\mathcal{Z}_{DT}\{x[k]\} + b\mathcal{Z}_{DT}\{y[k]\}
\]  
(14)
(3) Time-shifting

Consider an \( m \)-delayed signal \( x[k-m] \), whose DTFT is

\[
\mathcal{Z}_{DT} \{ x[k-m] \} = \sum_{k=-\infty}^{\infty} x[k-m] e^{-j\Omega k} = \sum_{p=-m}^{\infty} x[p] e^{-j(p+m)\Omega}
\]

(15)

\[
= \sum_{p=0}^{\infty} x[p] e^{-j(p+m)\Omega} = e^{-jm\Omega} \left( \sum_{p=0}^{\infty} x[p] e^{-j\Omega p} \right) = e^{-jm\Omega} X(\Omega)
\]

(4) Premultiplying \( e^{-j\Omega_0} \) to \( x[k] \)

\[
\mathcal{Z}_{DT} \{ e^{-j\Omega_0} x[k] \} = \sum_{k=-\infty}^{\infty} e^{-j\Omega_0} x[k] e^{-j\Omega k} = \sum_{k=0}^{\infty} x[k] e^{-j(k-\Omega - \Omega_0)}
\]

(16)

\[
= X(\Omega - \Omega_0)
\]

(5) Real \( x[k] \)

From the DTFT, we have

\[
X(\Omega) = Re[X(\Omega)] + j Im[X(\Omega)]
\]

(17)

where

\( Re[X(\Omega)] \) is even; \( Im[X(\Omega)] \) is odd;

\( |X(\Omega)| \) is even; \( \angle X(\Omega) \) is odd.

(6) Convolution in time

Define the convolution of \( x[k] \) and \( y[k] \) as below:

\[
x[k] \ast y[k] = \sum_{m=0}^{\infty} x[k-m] y[m]
\]

(18)

The DTFT of \( x[k] \ast y[k] \) is then obtained as

\[
\mathcal{Z}_{DT} \{ x[k] \ast y[k] \} = X(\Omega) Y(\Omega)
\]

(19)

(7) Convolution in frequency

The DTFT of \( x[k] \ast y[k] \) is obtained as

\[
\mathcal{Z}_{DT} \{ x[k] \ast y[k] \} = \frac{1}{2\pi} X(\Omega) \ast Y(\Omega)
\]

(20)

(8) Parseval’s Theorem

The Parseval’s theorem for discrete signals is given by

\[
\sum_{k=-\infty}^{\infty} |x[k]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\Omega)|^2 d\Omega
\]

(21)
Next, before we introduce the discrete Fourier transform, or DFT in short, let’s discuss the DTFT of periodic signals.

Consider a periodic signal \( x[k] \) with period \( N \) where \( N \) is an integer and a finite duration signal \( x_0[k] \) which is given as

\[
\begin{cases}
  x[k], & 0 \leq k \leq N-1 \\
  0, & \text{otherwise}
\end{cases}
\]  

(22)

The DTFT of \( x_0[k] \) is obtained as

\[
\Im \{ X_0(\Omega) \} = \sum_{k=0}^{N-1} x[k] e^{-jk\Omega}
\]

(23)

Then, the periodic signal can be expressed as

\[
x[k] = x_0[k] * \sum_{n=-\infty}^{\infty} \delta[k-nN]
\]

(24)

whose DTFT is

\[
X(\Omega) = X_0(\Omega) \sum_{n=-\infty}^{\infty} \Im \{ \delta[k-nN] \}
\]

(24)

\[
= X_0(\Omega) \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \delta[k-nN] e^{-jk\Omega} = X_0(\Omega) \sum_{n=-\infty}^{\infty} e^{-jnN\Omega}
\]

From the Fourier series of impulse train, we know that

\[
\delta_I(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT) = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{jk\frac{2\pi}{T}} = \sum_{n=-\infty}^{\infty} \frac{1}{T} e^{j\frac{2\pi}{N} nT}
\]

(25)

which implies

\[
\frac{N}{2\pi} \sum_{n=-\infty}^{\infty} e^{-jnN\Omega} = \sum_{n=-\infty}^{\infty} \delta(\Omega - n\frac{2\pi}{N})
\]

(26)

by changing \( t \) and \( T \) into \( \Omega \) and \( 2\pi/N \) respectively. Therefore, we have

\[
X(\Omega) = X_0(\Omega) \sum_{n=-\infty}^{\infty} e^{-jnN\Omega} = \frac{2\pi}{N} X_0(\Omega) \sum_{n=-\infty}^{\infty} \delta(\Omega - n\frac{2\pi}{N})
\]

(27)

\[
= \frac{2\pi}{N} \sum_{n=-\infty}^{\infty} X_0(\Omega) \delta(\Omega - n\frac{2\pi}{N})
\]

\[
= \frac{2\pi}{N} \sum_{n=-\infty}^{\infty} X_0(n\frac{2\pi}{N}) \delta(\Omega - n\frac{2\pi}{N})
\]

Since \( X(\Omega) \) is periodic with period \( 2\pi \), the \( N \) distinct \( x[k] \) for \( 0 \leq k \leq N-1 \) is transformed into \( N \) distinct \( X_0(2\pi n/N) \) for \( 0 \leq n \leq N-1 \). The IDTFT is then obtained as
\[ X[k] = \frac{1}{2\pi} \int_{0}^{2\pi} X(\Omega)e^{j\Omega k}d\Omega \]  

\[ = \frac{1}{N} \sum_{n=-\infty}^{\infty} X_0 \left( n \frac{2\pi}{N} \right) e^{j\frac{2\pi}{N} n k} \]

where

\[ X_0 \left( n \frac{2\pi}{N} \right) = \sum_{k=0}^{N-1} x[k] e^{-j\frac{2\pi}{N} n k} \]  

Clearly, there are \( N \) distinct values of \( x[k] \) and \( N \) distinct values of \( X_0(2n\pi/N) \). This is a very important observation. With these discrete values, we can easily calculate them in a digital way, called discrete Fourier transform or DFT in short, unlike the DTFT \( X(\Omega) \) which is continuous in frequency \( \Omega \).