15. LTI System in Difference Equation

In system engineering, an \(n\)th-order discretized LTI system can be described by the following advanced form:

\[
y[k + n] + a_{n-1}y[k + n - 1] + \cdots + a_1y[k + 1] + a_0y[k] = b_n u[k + n] + b_{n-1}u[k + n - 1] + \cdots + b_1u[k + 1] + b_0u[k]
\]

which in fact is equivalent to the following delayed form

\[
y[k] + a_{n-1}y[k - 1] + \cdots + a_1y[k - n + 1] + a_0y[k - n] = b_n u[k] + b_{n-1}u[k - 1] + \cdots + b_1u[k - n + 1] + b_0u[k - n]
\]

In this course we will focus on the delayed form.

Assume \(h[k]\) is the impulse response of a causal system and then \(h[k]=0\) for \(k<0\). It is known that for an initially relaxed system, i.e., \(y[k]=u[k]=0\) for \(k<0\), the input and output can be described by the following convolution

\[
y[k] = h[k] * u[k] = \sum_{m=0}^{k} h[k - m] u[m]
\]

which leads to

\[
Y(z) = H(z) U(z)
\]

where \(H(z)\) is the transfer function of the system. According to (2), taking z-transform yields

\[
Y(z) + a_{n-1}z^{-1}Y(z) + \cdots + a_1z^{-(n-1)}Y(z) + a_0z^{-n}Y(z) = b_n U(z) + b_{n-1}z^{-1}U(z) + \cdots + b_1z^{-(n-1)}U(z) + b_0z^{-n}U(z)
\]

and thus

\[
Y(z) = \frac{b_n + b_{n-1}z^{-1} + \cdots + b_1z^{-(n-1)} + b_0z^{-n}}{1 + a_{n-1}z^{-1} + \cdots + a_1z^{-(n-1)} + a_0z^{-n}} U(z)
\]

Compared to (4), it is clear that the transfer function is

\[
H(z) = \frac{Y(z)}{U(z)} = \frac{b_n + b_{n-1}z^{-1} + \cdots + b_1z^{-(n-1)} + b_0z^{-n}}{1 + a_{n-1}z^{-1} + \cdots + a_1z^{-(n-1)} + a_0z^{-n}}
\]

which is in negative-power form. We can rewrite (7) into a positive-power form as

\[
H(z) = \frac{b_n z^n + b_{n-1}z^{n-1} + \cdots + b_1z + b_0}{z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0}
\]

which is the form will be used in this course.
Similar to the Laplace transform, the discrete signals given as z-transform is also solved by using partial fraction expansion and then taking inverse z-transform. Here, we will adopt some examples for demonstration.

Example

What is the discrete signal $y[k]$ for $k \geq 0$ whose z-transform is

$$Y(z) = \frac{1}{(1 - z^{-1})(1 - 0.5z^{-1})}$$

Sol:

First, premultiply $Y(z)$ by $z^2$ and turn it into the positive-power form as

$$Y(z) = \frac{z^2}{(z-1)(z-0.5)}$$

Further express it as below:

$$Y(z) = \frac{z}{(z-1)(z-0.5)} = \frac{A_1}{z-1} + \frac{A_2}{z-0.5}$$

Hence, we have

$$z = A_1(z-0.5) + A_2(z-1)$$

and obtain $A_1 = 2$ and $A_2 = -1$. That results in

$$Y(z) = 2 \frac{z}{z-1} - 1 - \frac{z}{z-0.5}$$

Taking the inverse z-transform, the discrete signal $y[k]$ is

$$y[k] = 2 - (0.5)^k \quad \text{for} \ k \geq 0.$$ 

Example

An initially relaxed LTI discrete system is given as

$$y[k] - y[k-1] + 0.5y[k-2] = u[k] + u[k-1]$$

(A) What is the transfer function? Is the system stable?

(B) What is the impulse response?

(C) If $u[k]$ is a unit step input, then what is the output response $y[k]$?
Sol:

(A)
Taking the z-transform yields $Y(z) - z^{-1}Y(z) + 0.5z^{-2}Y(z) = U(z) + z^{-1}U(z)$, and then we obtain the transfer function as

$$H(z) = \frac{Y(z)}{U(z)} = \frac{1 + z^{-1}}{1 - z^{-1} + 0.5z^{-2}} = \frac{z^2 + z}{z^2 - z + 0.5} = \frac{z + 1}{(z - 0.707e^{-j45^\circ})(z - 0.707e^{j45^\circ})}$$

where the two poles are located in the unit circle and then the system is stable.

(B)
Further express $H(z)$ as below:

$$H(z) = \frac{z + 1}{z^2 - z + 0.5} = \frac{z + 1}{(z - 0.5 + j0.5)(z - 0.5 - j0.5)} = \frac{A}{z - 0.707e^{-j45^\circ}} + \frac{A^*}{z - 0.707e^{j45^\circ}}$$

Hence, we have

$$z + 1 = A(z - 0.5 - j0.5) + A^*(z - 0.5 + j0.5)$$

and obtain $A = 0.5 + j1.5 = \frac{\sqrt{10}}{2}e^{j71.6^\circ}$. That results in

$$H(z) = \frac{\sqrt{10}}{2}e^{j71.6^\circ} \frac{z}{z - 0.707e^{-j45^\circ}} + \frac{\sqrt{10}}{2}e^{-j71.6^\circ} \frac{z}{z - 0.707e^{j45^\circ}}$$

Taking the inverse z-transform, the impulse response $h[k]$ is

$$h[k] = \frac{\sqrt{10}}{2}e^{j71.6^\circ}(0.707e^{-j45^\circ})^k + \frac{\sqrt{10}}{2}e^{-j71.6^\circ}(0.707e^{j45^\circ})^k$$

$$= \frac{\sqrt{10}}{2}(0.707)^k e^{-j45^\circ(k-71.6^\circ)} + \frac{\sqrt{10}}{2}(0.707)^k e^{j45^\circ(k-71.6^\circ)}$$

$$= \sqrt{10}(0.707)^k \cos(45^\circ k - 71.6^\circ)$$

(C)
Since $U(z) = \frac{z}{z-1}$, we have

$$Y(z) = H(z)U(z) = \frac{z(z^2 + z)}{(z-1)(z^2 - z + 0.5)} = \frac{A_1z}{z-1} + \frac{A_2z}{z - 0.707e^{-j45^\circ}} + \frac{A_3z}{z - 0.707e^{j45^\circ}}$$
where \( A_1 = 4 \) and \( A_2 = 1.58e^{j161.6^\circ} \). Hence, we have
\[
y[k] = 4 + 3.16(0.707)^k \cos(45^\circ k - 161.6^\circ)
\]

**Exercise:**

An initially relaxed LTI discrete system is given as
\[
y[k] + 0.1y[k-1] + 0.2y[k-2] = u[k] + u[k-1]
\]

(A) What is the transfer function? Is the system stable?
(B) What is the impulse response?
(C) If \( u[k] \) is a unit step input, then what is the output response \( y[k] \)?