I. **Analytic function** (also called **regular, holomorphic function**)

\[ w = f(z), \quad z = x + iy \rightarrow w = u(x, y) + iv(x, y) \]

**Limit:** 
\[ \lim_{z \to z_0} f(z) = l \iff \forall \varepsilon > 0, \exists \delta, \text{ s.t. } |z - z_0| < \delta \]

**Def. Analyticity**

A function \( f(z) \) is said to be **analytic** in a **domain** \( R \) if it is defined and **differentiable** at all points of \( R \). The function \( f(z) \) is said to be analytic at a **point** \( z = z_0 \) in \( R \) if \( f(z) \) is analytic in a **neighborhood** of \( z_0 \). Then,

\[
 f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}
\]

at a point and its neighborhood must exist no matter how \( \Delta z \) approaches zero.

---

*Fig. 3.30. Limit*
Examples

Polynomial function:

\[ f(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n, \quad n \geq 0 \]

is analytic in the entire complex \( z \)-plane.

Rational function:

\[ f(z) = \frac{g(z)}{h(z)}, \]

where \( g(z) \) and \( h(z) \) are coprime polynomials. This function is analytic in the complex plane except at the singular points where \( h(z) = 0 \).
If $f(z)$ is analytic, then $f'(z) = \frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$ at a point and its neighborhood must exist no matter how $\Delta z$ approaches zero.

\[
\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \\
= \lim_{\Delta x \to 0} \lim_{\Delta y \to 0} \frac{[u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y)] - [u(x, y) + iv(x, y)]}{\Delta x + i\Delta y}
\]
(1) If $\Delta x \neq 0, \Delta y = 0 \leftarrow \frac{\Delta Z}{\Delta x}$

$$\frac{dw}{dz} = \lim_{\Delta x \to 0} \frac{[u(x+\Delta x, y) + iv(x+\Delta x, y)] - [u(x, y) + iv(x, y)]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[ \frac{u(x+\Delta x, y) - u(x, y)}{\Delta x} + i \frac{v(x+\Delta x, y) - v(x, y)}{\Delta x} \right] = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

(2) If $\Delta x = 0, \Delta y \neq 0 \downarrow \Delta Z$

$$\frac{dw}{dz} = \lim_{\Delta y \to 0} \frac{[u(x, y+\Delta y) + iv(x, y+\Delta y)] - [u(x, y) + iv(x, y)]}{i \Delta y}$$

$$= \lim_{\Delta y \to 0} \left[ \frac{u(x, y+\Delta y) - u(x, y)}{i \Delta y} + i \frac{v(x, y+\Delta y) - v(x, y)}{i \Delta y} \right] = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

(1) $= (2): \quad \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$

$$\begin{vmatrix}
\frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} & -\frac{\partial v}{\partial x}
\end{vmatrix}$$

Cauchy-Riemann Condition

$\rightarrow$ Necessary and sufficient condition for $f(z)$ to be analytic
Property 1.

If \( u, v \in C^2 \), \( f(z) = u + iv \) is **analytic**, then both \( u \) and \( v \) satisfy the 2-D Laplace equation. (\( u \): harmonic function, \( v \): conjugate harmonic function)

\[
\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0
\]

pf:

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}
\]

\[
\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}
\]

\[
\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0
\]

Similarly, \( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \) \hspace{1cm} \text{Q.E.D.}

This reveals that analytic functions can be closely related to conservative/potential fields.
Property 2.

If \( w = u(x, y) + iv(x, y) \) is analytic, then \( u(x, y) = c \perp v(x, y) = k \)

pf:

\[
\begin{align*}
\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy &= 0 \\
\Rightarrow \left( \frac{dy}{dx} \right)_u &= -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy &= 0 \\
\Rightarrow \left( \frac{dy}{dx} \right)_v &= -\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} = \frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}}
\end{align*}
\]

(Cauchy-Riemann condition)

Thus, \( \left( \frac{dy}{dx} \right)_u \left( \frac{dy}{dx} \right)_v = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} \cdot \frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}} = -1 \)

i.e., \( u = c \perp v = k \)  Q.E.D.
Property 3.

If in any analytic function \( w = u(x, y) + iv(x, y) \), \( x \) and \( y \) are replaced by

\[
x = \frac{z + \bar{z}}{2} \quad \text{and} \quad y = \frac{z - \bar{z}}{2i},
\]
i.e., \( z = x + iy \), then \( \frac{\partial w}{\partial \bar{z}} \equiv 0 \)

\[
\text{pf:} \quad \frac{\partial w}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}} (u + iv) = \frac{\partial u}{\partial \bar{z}} + i \frac{\partial v}{\partial \bar{z}} = \left( \frac{\partial u}{\partial x} \frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \bar{z}} \right) + i \left( \frac{\partial v}{\partial x} \frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial \bar{z}} \right)
\]

But \( \frac{\partial x}{\partial \bar{z}} = \frac{1}{2}, \quad \frac{\partial y}{\partial \bar{z}} = -\frac{1}{2i} = \frac{i}{2} \)

\[
\frac{\partial w}{\partial \bar{z}} = \left( \frac{1}{2} \frac{\partial u}{\partial x} + \frac{i}{2} \frac{\partial u}{\partial y} \right) + i \left( \frac{1}{2} \frac{\partial v}{\partial x} + \frac{i}{2} \frac{\partial v}{\partial y} \right) = \frac{1}{2} \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) + \frac{i}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)
\]

\[
= 0 + i0 \equiv 0 \quad (\because \text{Cauchy-Riemann condition})
\]

Def

If \( f(z) \) is not analytic at \( z_0 \), but if every neighborhood of \( z_0 \) contains points at which \( f(z) \) is analytic, then \( z_0 \) is called a singular point of \( f(z) \).
Elementary Function

* \( e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x (\cos y + i \sin y) \),

\[
\frac{de^z}{dz} = e^z
\]

* \( \cos z = \frac{e^{iz} + e^{-iz}}{2} \), \( \sin z = \frac{e^{iz} - e^{-iz}}{2i} \)

\[
\cos z = \cos(x+iy) = \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} = \frac{e^{-y}e^{ix} + e^ye^{-ix}}{2} = \frac{e^{-y}(\cos x + i \sin x) + e^y(\cos x - i \sin x)}{2}
\]

\[
= \cos x \frac{e^y + e^{-y}}{2} - i \sin x \frac{e^y - e^{-y}}{2} = \cos x \cosh y - i \sin x \sinh y
\]

Similarly, \( \sin z = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y \)

If \( x = 0 \), then \( \cos iy = \cosh y \), \( \sin iy = i \sinh y \)
* $w = \ln z$:

$$w = \ln \{|z| e^{i\theta}\} = \ln |z| + i\theta = \ln |z| + i(\theta_p + 2\pi k)$$

$$= \text{Ln } z + 2\pi ki, \quad 0 \leq \theta_p < 2\pi, k = 0, \pm 1, \pm 2, \ldots$$

When $k = 0$, $w$ has the "principal value" $w = \text{Ln } z = \ln |z| + i\theta_p$

Thus, $w = \ln z$ is a "multivalued" function. (infinitely many branches)
To restrict the function to be a single-valued function, use a "branch cut" to enable the transition from a branch to another branch. The choice of branch cut is not unique. \( \Rightarrow \) Riemann surface

The multi-valued function \( f(z) \) is discontinuous along the branch cut.
Note:
1. The function is discontinuous along the branch cut (singularity).
2. A branch cut must connect pairs of branch point (including $\infty$).
3. A branch cut prevent the complete encirclement of a branch point such that single-valued analyticity is maintained.
4. The choice of branch cut is not unique.
5. For an "elementary function" to have branch point, they must be
   (1) $\ln z$ or (2) $z^c$, $c$: noninteger
   $\rightarrow e^{\bar{z}}$ is an 'entire function' that is analytic for all $z$, while $\ln z$ is not.
General powers

$$z^c = e^{c \ln z}$$

If \( c \) is an integer, then \( z^c \) is single-valued.
If \( c = 1/n, \ n = 2, 3, \ldots \), then \( z^c \) is multi-valued, i.e.,

$$z^c = \sqrt[n]{z} = e^{(1/n) \ln z}, \quad z \neq 0$$

The exponential is determined up to multiples of \( 2\pi i / n \) and we obtain the \( n \) distinct values of the \( n \)th root.

- **multiple values**
  - need to define principal value, branch-cut, Riemann surface

(p.746, Kreyszig)
The double-valued relation $w = \sqrt{z}$ becomes single-valued on the Riemann surface.

The upper sheet of the Riemann surface: the principal value $\rightarrow$ mapped to the right $w$-half-plane.

The lower sheet of the Riemann surface: other value $\rightarrow$ mapped to the left $w$-half-plane. $z = 0$ is the branch point.

- The infinitely many-valued natural logarithm
  
  $w = \ln z = \text{Ln } z + 2n\pi i \quad n = 0, \pm 1, \pm 2, \cdots$

  becomes single-valued on a Riemann surface consisting of infinitely many sheets. The principal value $w = \text{Ln } z$ maps its sheet onto the horizontal strip $-\pi < v \leq \pi$. The function $w = \ln z + 2\pi i$ maps its sheet onto the neighboring strip $\pi < v \leq 3\pi$ and so on. The mapping of the point $z \neq 0$ of the Riemann surface onto the points of the $w$-plane is one-to-one.
Ex: \[ f(z) = (z^2 - 1)^{1/2} = (z + 1)^{1/2}(z - 1)^{1/2} \]

sol:

note that \( z^a = e^{a \ln z} \), \( z = 0 \)

is the branch pt.

\( (z + 1)^{1/2} \) has a branch pt. at \( z = -1 \)

\( (z - 1)^{1/2} \) has a branch pt. at \( z = 1 \)

Let \( z + 1 = re^{i\theta} \), \( z - 1 = \rho e^{i\phi} \)

Then \( f(z) = \sqrt{r \rho} e^{i(\theta + \phi)/2} \)

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<th>( \phi )</th>
<th>( (\theta + \phi)/2 )</th>
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<td>0</td>
<td>0</td>
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<td>( \pi/2 )</td>
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<td>( \pi )</td>
<td>( 3\pi/2 )</td>
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<td>8</td>
<td>( 2\pi )</td>
<td>( 2\pi )</td>
<td>( 2\pi )</td>
</tr>
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</table>
$w = \cos^{-1} z$

$z = \cos w = \frac{e^{iw} + e^{-iw}}{2}$

$e^{2iw} - 2ze^{iw} + 1 = 0$

$e^{iw} = z \pm z\sqrt{z^2 - 1} \Rightarrow iw = \ln(z \pm \sqrt{z^2 - 1})$

$w = \cos^{-1} z = -i \ln(z \pm \sqrt{z^2 - 1})$

(multivalued!)

Similar for $\sin^{-1} z$ and $\tan^{-1} z$, etc.
\[ \tan^{-1} \]

\[
\begin{align*}
\text{Diagram 1:} & \quad +i \\
\text{Diagram 2:} & \quad -i
\end{align*}
\]
General Properties:
1. The branch cut should start from branch points which are singular points on the complex plane ($\infty$ is considered a singular pt.)
2. Branch points always occur in pairs. Branch lines must connect pairs of branch points.
3. Encirclement of a branch point causes us to move from one branch to another. The branch cut is regarded as a barrier to encirclement of the branch point such that single-valued analyticity is maintained.
4. $\Delta\phi$ at two sides of a branch cut is $2\pi/N$, $N$ is the root power.
5. The choice of branch cuts is not unique. The branch cuts need not be straight lines. However, the branch cuts are required to cause as little interference to analytic continuation as possible. In some cases, the choice of branch cuts depend upon physical considerations.
6. In some case, branch cuts merge to yield analytic region.

7. Elementary functions that have branch pt.: \( \ln z \) or \( z^a \), a noninteger
X Example (p. 242, Junger & Feit)

\[ \gamma - \text{plane} \]

\[ -\frac{\pi}{2} \leq \phi \leq 0 \]

\[ 0 \leq \phi \leq \frac{\pi}{2} \]

\[ 0 \leq \phi \leq \frac{\pi}{2} \]

\[ -\frac{\pi}{2} \leq \phi \leq 0 \]

\[ Mingsian R. Bai \]
Requirement:
Decaying waves propagating in +z direction
\[ p(z,t) = e^{i(k_z - \omega t)}, \]

where \( k_z = \sqrt{k^2 - \gamma^2} = i\sqrt{\gamma - k}\sqrt{\gamma + k} = i(\alpha - i\beta) = \beta + i\alpha \)
\( \alpha, \beta > 0 \)

Thus, \( p(z,t) = e^{i[(\beta + i\alpha)z - \omega t]} = e^{-\alpha z} \cdot e^{i(\beta z - \omega t)}\)

Let \( \phi_1 = \angle(\gamma - k) \)
\( \phi_2 = \angle(\gamma + k) \)
\[ \phi = \frac{1}{2}(\phi_1 + \phi_2) = \angle\sqrt{\gamma^2 - k^2} \]
<table>
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<th>$\phi_2$</th>
<th>$\phi = \frac{1}{2}(\phi_1 + \phi_2)$</th>
<th>Re</th>
<th>i*Im</th>
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<td>$-\frac{\pi}{2}$</td>
<td>$-\frac{\pi}{2}$</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>

**Angle Discussion**
For example

Point 1:

\[ \phi_1 = \pi \]

\[ \phi_2 = 0 \]

Point 11:

\[ \phi_1 = -\frac{\pi}{2} \]

\[ \phi_2 = \frac{3\pi}{2} \]
Integration in the complex plane

\[ \int_C f(z)dz = \lim_{n \to \infty} \sum_{k=1}^{n} f(\zeta_k)\Delta z_k \]

If \( C \) is a closed curve,
\[ \oint_C f(z)dz \]
is a contour integral.
\( \oint_C \frac{dz}{(z-z_0)^n+1} = \int_0^{2\pi} \frac{d(re^{i\theta})}{(re^{i\theta})^{n+1}} = \int_0^{2\pi} \frac{rie^{i\theta}d\theta}{r^{n+1}e^{i(n+1)\theta}} = \frac{i}{r^n} \int_0^{2\pi} e^{-in\theta}d\theta \)

If \( n = 0 \), \( \oint_C \frac{dz}{z-z_0} = i \int_0^{2\pi} d\theta = 2\pi i \)

If \( n \neq 0 \), \( \frac{i}{r^n} \int_0^{2\pi} (\cos n\theta - i \sin n\theta)d\theta = \frac{i}{r^n} \left( \frac{\sin n\theta}{n} + i \frac{\cos n\theta}{n} \right) \bigg|_0^{2\pi} = 0 \)
Recall the theorems in vector analysis.

**Thm 1**  Green's theorem
\[ \oint_C (P \, dx + Q \, dy) = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dxdy \]  (proved by Stoke's theorem)

**Thm 2**  Potential theory (conservative field)

If \( \phi = \int \mathbf{F} \cdot d\mathbf{r} = \int P \, dx + Q \, dy \) is independent of the integration path,
where \( \mathbf{F} = P\mathbf{i} + Q\mathbf{j} \) and \( \mathbf{r} = x\mathbf{i} + y\mathbf{j} \),

then there exists \( \phi \) such that \( \mathbf{F} = \nabla \phi \) i.e., \( \frac{\partial \phi}{\partial x} = P(x, y), \quad \frac{\partial \phi}{\partial y} = Q(x, y) \).

**Thm 3**  From Thm 1, if \( \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \)
\[ \iff \oint_C P \, dx + Q \, dy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dxdy = 0 \iff \int_A P \, dx + Q \, dy \]

is independent of the path.
**Thm** Cauchy's theorem

If $R$ is a simply connected region, whose boundary is sectionally smooth, and if $f(z)$ is analytic and $f'(z)$ is continuous in $R$, then

$$\oint_C f(z)\,dz = 0$$

where $C$ is the entire boundary of $R$.

pf: $\oint_C f(z)\,dz = \oint_C (u + iv)d(x + iy) = \oint_C (u + iv)(dx + idy)$

$$= \oint_C (udx - vdy) + i\oint_C (vdx + udy)$$

$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\,dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\,dx\,dy \quad \text{(by Green's theorem)}$$

$$= 0 + i0 = 0 \quad \text{ (Cauchy-Riemann condition)} \quad \text{Q.E.D.}$$
**Principle of path deformation**

\[ (\oint_{C_1} + \oint_{C_2} + \int_{A}^{B} + \int_{B}^{A}) f(z) dz = 0 \]

\[ (\oint_{C_1} + \oint_{C_2}) f(z) = 0 \]

\[ \oint_{C_1} f(z) dz = \oint_{C_2} f(z) dz \]

\[ \downarrow \]

A simple one in general
\[ \oint f(z)dz = 0 \Rightarrow \left( \int_{c_1} - \int_{c_2} \right) f(z)dz = 0 \]
\[ \Rightarrow \int_{c_1} f(z)dz = \int_{c_2} f(z)dz \]
Independent of the path!

**Thm 9**
If \( \nabla^2 u = 0 \) (\( u \) is called a harmonic function), then there exists an analytic function \( f(z) \) such that
\[
f(z) = u + iv \quad \text{with} \quad u = c \perp v = k
\]
where
\[
v = \int_{(a,b)}^{(x,y)} \frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy = \int_{(a,b)}^{(x,y)} P(x, y)dx + Q(x, y)dy
\]
\[
P(x, y) = -\frac{\partial u}{\partial y}, \quad Q(x, y) = \frac{\partial u}{\partial x}
\]
pf:

\[ \nabla^2 u = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial y} \left( -\frac{\partial u}{\partial y} \right) \Rightarrow \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \]

\[ \Rightarrow \oint Pdx + Qdy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = 0 \quad \text{(Green's theorem)} \]

or \[ \int_Pdx + Qdy \text{ is independent of the path} \quad (\because \text{Thm 3}), \text{potential field!} \]

Construct a potential function: \( v = \int_{(a,b)}^{(x,y)} P(x, y)dx + Q(x, y)dy \) such that

\[ \nabla v = Pi + Qj \]

\[ \Rightarrow \frac{\partial v}{\partial x} = P = -\frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial y} = Q = \frac{\partial u}{\partial x} \quad (\because \text{Thm 2}) \]

These are precisely the Cauchy-Riemann conditions which are the conditions that \( f(z) = u + iv \) be an analytic function.
Thm 10  Cauchy's integral formula

If \( f(z) \) is analytic within and on the boundary \( C \) of a simply connected region \( R \) whose boundary is sectionally smooth, and if \( z_0 \) is any point in the interior of \( R \), then

\[
f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0} \, dz
\]

(value at an interior point can be determined by the BC)

pf: Since \( \frac{f(z)}{z-z_0} \) is analytic everywhere in \( R \) except at \( z = z_0 \),

\[
\frac{f(z)}{z-z_0}
\]

is analytic everywhere in \( R' \), i.e., \( z = z_0 \) is the only singularity.
Hence, \( \oint_{c} \frac{f(z)}{z-z_0} \, dz = \oint_{c_0} \frac{f(z)}{z-z_0} \, dz = \oint_{c_0} \frac{f(z_0) + [f(z) - f(z_0)]}{z-z_0} \, dz \)

\[
= f(z_0) \oint_{c_0} \frac{dz}{z-z_0} + \oint_{c_0} \frac{f(z)-f(z_0)}{z-z_0} \, dz
\]

\[
\downarrow
\]

\[
\oint_{c_0} \frac{dz}{z-z_0} = 2\pi i \quad \text{(Ex1)}
\]

On the other hand, \( \left| \oint_{c_0} \frac{f(z)-f(z_0)}{z-z_0} \, dz \right| \leq \oint_{c_0} \left| \frac{f(z)-f(z_0)}{z-z_0} \right| \, |dz| \)

Since \( f(z) \) is analytic and continuous, \( \forall \varepsilon > 0, \exists \delta \) such that

\[
|f(z) - f(z_0)| < \varepsilon, \quad \forall |z-z_0| \equiv \rho < \delta
\]

\[
\left| \oint_{c_0} \frac{f(z)-f(z_0)}{z-z_0} \, dz \right| < \oint_{c_0} \frac{\varepsilon}{\rho} \, |dz| = \frac{\varepsilon}{\rho} \oint_{c_0} |dz| = \frac{\varepsilon}{\rho} \cdot 2\pi\rho = 2\pi\varepsilon \to 0
\]
Thus, $\oint_{C_0} \frac{f(z)}{z-z_0} \, dz = f(z_0)(2\pi i) + 0 \Rightarrow f(z_0) = \frac{1}{2\pi i} \oint_{C_0} \frac{f(z)}{z-z_0} \, dz$

Q.E.D.

* If $C$ is a circle, $z - z_0 = re^{i\theta}$, $r$ is a constant.

$$f(z_0) = \frac{1}{2\pi i} \oint_{C} \frac{f(z) \, dz}{z - z_0} = \frac{1}{2\pi i} \int_{0}^{2\pi} \frac{f(z)}{re^{i\theta}} r e^{i\theta} \, d\theta = \frac{1}{2\pi} \int_{0}^{2\pi} f(z) \, d\theta$$

$$= \frac{1}{2\pi r} \int_{0}^{2\pi r} f(z) \, ds \rightarrow \text{Gauss mean value theorem}$$

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In addition, $f'(z_0) = \lim_{\Delta z_0 \to 0} \frac{f(z_0 + \Delta z_0) - f(z_0)}{\Delta z_0}$

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz$$

$$= \lim_{\Delta z_0 \to 0} \frac{1}{\Delta z_0} \left[ \frac{1}{2\pi i} \oint_C \frac{f(z)dz}{z - (z_0 + \Delta z_0)} - \frac{1}{2\pi i} \oint_C \frac{f(z)dz}{z - z_0} \right]$$

$$= \lim_{\Delta z_0 \to 0} \frac{1}{\Delta z_0} \left\{ \frac{1}{2\pi i} \oint_C f(z) \left[ \frac{1}{z - (z_0 + \Delta z_0)} - \frac{1}{z - z_0} \right] dz \right\}$$

$$= \lim_{\Delta z_0 \to 0} \frac{1}{\Delta z_0} \left\{ \frac{1}{2\pi i} \oint_C f(z) \left[ \frac{\Delta z_0}{(z - z_0 - \Delta z_0)(z - z_0)} \right] dz \right\}$$

$$= \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2} dz$$

Thus, by induction, we have the following theorem.
Thm 11
If \( f(z) \) is analytic in a closed simply connected region \( R \), then at any interior point \( z_0 \) of \( R \), the derivatives of \( f(z) \) of all orders exist and are analytic.

\[
f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)dz}{(z-z_0)^{n+1}}
\]

→ A major difference between complex variables and real variables

(1) In complex variables, \( f'(z_0) \) exists \( \Rightarrow f(z_0) \) is analytic in the neighborhood of \( z = z_0 \) \( \Rightarrow f^{(n)}(z_0) \) exists for all \( n > 0 \).
(2) In real variables, 

\[ f'(x) \text{ exists does not always imply that } f^{(n)}(x) \text{ exists for all } n > 0 \]

Ex: \[ f(x) = x^{7/3}, \quad f'(x) = \frac{7}{3} x^{4/3}, \quad f''(x) = \frac{7 \cdot 4}{3 \cdot 3} x^{1/3} \]

\[ f'''(x) = \frac{7 \cdot 4 \cdot 1}{3 \cdot 3 \cdot 3} x^{-2/3} \text{ does not exist at } x = 0 \]

Thm 13 Cauchy's inequality

If \( f(z) \) is analytic within and on a circle \( C \) of radius \( r \) with center at \( z_0 \), then \( \left| f^{(n)}(z_0) \right| \leq \frac{n!M}{r^n} \) where \( M \) is the maximum of \( |f(z)| \) on \( C \).

pf: \[
\left| f^{(n)}(z_0) \right| = \left| \frac{n!}{2\pi i} \oint_C \frac{f(z)dz}{(z-z_0)^{n+1}} \right| \leq \frac{n!}{2\pi} \oint_C \frac{|f(z)||dz|}{|z-z_0|^{n+1}} \\
\leq \frac{n!}{2\pi} \frac{M}{r^{n+1}} \oint_C |dz| = \frac{n!}{2\pi} \frac{M}{r^{n+1}} 2\pi r = \frac{n!M}{r^n}
\]
**Maximum value theorem**  
Special case for $n = 0$:  
$$|f(z_0)| \leq M$$

The maximum value of $|f(z)|$ cannot occur in the interior of $R$; it must occur on the boundary $C$.

→ A similar statement can be obtained for the minimum value by applying the above theorem to $|1/f(z)|$.

Ex:  
$$\oint_C \frac{e^z}{z^2 + 1} \, dz = ?$$  
if  
(a) $C: |z - i| = 1$,  
(b) $C: |z + i| = 1$

Sol:  
(a)  
$$\oint_C \left( \frac{e^z}{z + i} \right) \frac{dz}{z - i} = \oint_C f(z) \frac{dz}{z - i} = 2\pi i f(i)$$  
$$= 2\pi i \frac{e^i}{2i} = \pi e^i = \pi (\cos 1 + i \sin 1)$$
(b) \[ \oint_C \left( \frac{e^z}{z-i} \right) \frac{dz}{z-(-i)} = \oint_C f(z) \frac{dz}{z-(-i)} = 2\pi i f(-i) \]
\[ = 2\pi i \frac{e^{-i}}{-2i} = -\pi e^{-i} = -\pi (\cos 1 - i \sin 1) \]

Morera Theorem
\[ \oint_C f(z)dz = 0 \quad \forall C \text{ in } R \Rightarrow f(z) \text{ is analytic in } R \]

2-D Potential Flow

\[ \vec{V} = \nabla \phi \]

\[ \nabla \times \vec{V} = 0 \rightarrow \text{Conservative field} \rightarrow \text{Potential function } \phi \]
\[ \rightarrow \text{Path independence} \rightarrow \text{Laplace equation} \]

Potential flow: inviscid \((\nabla \times \vec{V} = 0)\) and incompressible flow \((\nabla \cdot \vec{V} = 0)\)

\[ \nabla \times \vec{V} = 0 \Rightarrow \exists \phi \quad \vec{V} = \nabla \phi \Rightarrow \nabla \cdot (\nabla \phi) = \nabla^2 \phi = 0 \]
Define an analytic complex potential

\[ \Phi(z) = \phi(x, y) + i\psi(x, y) \]

where \( \phi(x, y) \): velocity potential, \( \psi(x, y) \): stream function

\[ V_x = \frac{\partial \phi}{\partial x}, \quad V_y = \frac{\partial \phi}{\partial y} \Rightarrow (\nabla = \nabla \phi) \perp (\phi = c) \]

Since \( (\phi = c) \perp (\psi = k) \) for analytic \( \Phi(z) \Rightarrow \vec{V} \parallel (\psi = k) \)
From Cauchy-Riemann condition,
\[
\frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y} = -V_y, \quad \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} = V_x
\]

\[\Rightarrow V_x = \frac{\partial \psi}{\partial y}, \quad V_y = -\frac{\partial \psi}{\partial x}\]

\[d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = -V_y dx + V_x dy\]

\[= (V_x \hat{i} + V_y \hat{j}) \cdot \left( \frac{dy}{ds} \hat{i} - \frac{dx}{ds} \hat{j} \right) ds\]

\[= \vec{V} \cdot \hat{n} ds = d(\text{flux})\]
The function \( \psi \) is termed the \textbf{stream function}. (path independent)

Since \( \Phi(z) \) is analytic, the derivative can be taken along any \((x)\) direction,

\[
\Phi'(z) = \frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial x} = V_x - iV_y
\]

\( \Rightarrow \Phi'(z) = V_x + iV_y = \vec{V} \) (complex velocity)

\[
\oint_C \Phi'(z)dz = \oint_C (V_x - iV_y)(dx +idy)
\]

\[
= \oint_C (V_xdx + V_ydy) + i(V_xdy - iV_ydx)
\]

\[
= \oint_C \left( \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy \right) + i \left( \frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial x} dx \right)
\]

\[
= \oint_C (d\phi + id\psi) = \oint_C d\phi + i \oint_C d\psi = \oint_C (\vec{V} \cdot \hat{u} + i\vec{V} \cdot \hat{n})ds
\]

\( = \text{circulation} + i \cdot \text{flux} \)

\( \hat{u} \): unit tangential vector, \( \hat{n} \): unit normal vector

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Ex:  

1. Uniform flow  
\[ \Phi(z) = Uz = U(x + iy) \Rightarrow \phi = Ux, \ \psi = Uy \]  
\[ \Phi'(z) = U = V_x + iV_y \Rightarrow V_x = U, \ \ \ V_y = 0 \]

2. Point source  
\[ \Phi(z) = \frac{A}{2\pi} \ln z = \frac{A}{2\pi} \ln(\rho e^{i\theta}) = \frac{A}{2\pi} (\ln \rho + i\theta) \Rightarrow \phi = \frac{A}{2\pi} \ln \rho, \psi = \frac{A}{2\pi} \theta \]  
\[ \mathbf{V} = \Phi'(z) = \frac{A}{2\pi} \frac{1}{z} = \frac{A}{2\pi} e^{i\theta} \]

Flux:  
\[ \psi_C = \oint_C d\psi = \int_0^{2\pi} \frac{A}{2\pi} d\theta = A, \quad (\text{C is a circle}) \]

Circulation:  
\[ \phi_C = \oint_C d\phi = \frac{A}{2\pi} \oint_C d(\ln \rho) = 0 \]

3. Point vortex  
\[ \Phi(z) = -\frac{iK}{2\pi} \ln z = -\frac{iK}{2\pi} \ln(\rho e^{i\theta}) \Rightarrow \phi = \frac{K \theta}{2\pi}, \psi = -\frac{K}{2\pi} \ln \rho \]
Poisson's integral formula for a circle (Dirichlet problem)
→ solving boundary value problems involving 2D Laplace equations

\[ \nabla^2 u = 0 \]

st.
\[ u = u(R, \phi) \]
given on a circle \( C \).
( Dirichlet problem)
Let $t = Re^{i\phi}$ be the source point and $z = re^{i\theta}$ be the field point. Let $f(z)$ be analytic in and on the circle $C$ whose center is the origin and whose radius is $R > r$. (interior problem)

By Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(t)dt}{t-z} \quad (1)$$

Now, let $z_1 = \frac{tt}{z} = \frac{R^2}{r} e^{i\theta}$ be the 'image point'. Since $R > r$, $z_1$ lies outside the circle $C$. Hence,

$$0 = \frac{1}{2\pi i} \oint_C \frac{f(t)dt}{t-z_1} \quad (2)$$

($\because t-z_1 \neq 0 \therefore$ The integrand is analytic inside and on $C$)
(1) – (2):

\[ f(z) = \frac{1}{2\pi i} \oint_C f(t) \left( \frac{1}{t-z} - \frac{1}{t-z_1} \right) dt = \frac{1}{2\pi i} \oint_C f(t) \left( \frac{1}{t-z} - \frac{\bar{z}}{\bar{z}t-\bar{t}} \right) dt \]

\[ = \frac{1}{2\pi i} \oint_C f(t) \left( \frac{\bar{t}t - \bar{z}\bar{z}}{(t-z)(\bar{t} - \bar{z})} \right) \frac{dt}{t} = \frac{1}{2\pi i} \oint_C f(t) \left( \frac{R^2 - r^2}{(t-z)(\bar{t} - \bar{z})} \right) \frac{dt}{t} \]

\[ = \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\phi}) \frac{(R^2 - r^2)d\phi}{(Re^{i\phi} - re^{i\theta})(Re^{-i\phi} - re^{-i\theta})} \]

\[ \therefore dt = Rei\phi d\phi = i\theta d\phi \]

\[ = \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\phi}) \frac{(R^2 - r^2)d\phi}{R^2 - 2Rr \cos(\phi - \theta) + r^2} = f(re^{i\theta}) = u(r, \theta) + iv(r, \theta) \]

\[ \Rightarrow u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} u(R, \phi) \frac{(R^2 - r^2)d\phi}{R^2 - 2Rr \cos(\phi - \theta) + r^2}, \quad r < R \]

\[ \text{cosine thm: distance between the source & field pts} \]
Poisson's integral formula for a half plane (Dirichlet problem)

Let \( f(w) = \phi(u, v) + i\psi(u, v) \) be an analytic function for \( v \geq 0 \).

Assume \( |z^k f(z)| < M \), \( \text{Im}(z) \geq 0 \), \( k > 0 \)

Cauchy integral formula:

\[
f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w) \, dw}{w - z}
\]

(source pt)\( \text{---(1)} \)

\[
0 = \frac{1}{2\pi i} \oint_C \frac{f(w) \, dw}{w - \bar{z}}
\]

(image pt)\( \text{---(2)} \)

\( \nabla^2 \phi = 0 \)

\( st. \)

\( \phi = \phi(u, 0) \) given on the \( u \)-axis.

(Dirichlet problem)
(1) - (2):
\[
f(z) = \frac{1}{2\pi i} \oint_C f(w) \left( \frac{1}{w-z} - \frac{1}{w-\overline{z}} \right) \, dw = \frac{1}{2\pi i} \oint_C \frac{(z-\overline{z})}{(w-z)(w-\overline{z})} f(w) \, dw
\]
Let \( z = x + iy \) and \( w = u + iv \). \( C_R \) is a very large semicircle.
\[
f(z) = \frac{1}{2\pi i} \oint_C \frac{(z-\overline{z}) f(w) \, dw}{(w-z)(w-\overline{z})}
\]
\[
= \frac{1}{2\pi i} \int_{-R}^{R} \frac{2iyf(u) \, du}{(u-x)^2 + y^2} + \frac{1}{2\pi i} \int_{C_R} \frac{2iyf(w) \, dw}{(w-z)(w-\overline{z})}
\]
\[
= \frac{y}{\pi} \int_{-R}^{R} \frac{f(u) \, du}{(u-x)^2 + y^2} + \frac{y}{\pi} \int_{C_R} \frac{f(w) \, dw}{(w-z)(w-\overline{z})}
\]
\[
|w^k f(w)| < M, \quad \text{Im}(w) \geq 0, \quad k > 0
\]
\[
\Rightarrow \left| \frac{f(w)}{(w-z)(w-\overline{z})} \right| = \left| \frac{w^{1-k} w^k f(w)}{(w-z)(w-\overline{z})} \right| < \frac{M}{R^{k+1}} \to 0 \quad \text{as} \quad R \to \infty
\]
With the result above and Thm 1 of limiting contour (will be shown later),

\[
\lim_{R \to \infty} \int_{C_R} \frac{f(w) \, dw}{(w - z)(w - \bar{z})} = 0
\]

Hence, \( f(z) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(u) \, du}{(u - x)^2 + y^2} \)

Let \( f(z) = \phi(x, y) + i\psi(x, y) \) and \( f(w) = \phi(u, v) + i\psi(u, v) \)

\[
\phi(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\phi(u, 0) \, du}{(u - x)^2 + y^2}, \quad \psi(x, y) = \int_{-\infty}^{\infty} \frac{\psi(u, 0) \, du}{(u - x)^2 + y^2}
\]

- Poisson's integral formula can also be obtained by Green's function method.
Infinite Series in the complex plane

\[ S(z) = f_1(z) + f_2(z) + \cdots + f_n(z) + \cdots \]

The partial sum: \( S_n(z) = f_1(z) + \cdots + f_n(z) \)

**Convergent:** \( \forall \varepsilon > 0, \exists N(\varepsilon, z) \text{ s.t. } |S(z) - S_n(z)| < \varepsilon, \forall n > N \)

Absolutely convergent: \( |f_1(z)| + \cdots + |f_n(z)| + \cdots \) is convergent

Absolute convergence \( (\sum_{n=0}^{\infty} |f_n(z)|) \Rightarrow \text{Convergence} \ (\sum_{n=0}^{\infty} f_n(z)) \)
Thm 2  Ratio test

For the series $f_1(z) + f_2(z) + \cdots + f_n(z) + \cdots$

Let $\lim_{n \to \infty} \left| \frac{f_{n+1}(z)}{f_n(z)} \right| = |r(z)|$

(1) diverges if $|r(z)| > 1$

(2) converges if $1 > |r(z)| \geq 0$

(3) undetermined if $|r(z)| = 1$

Def. Uniform convergence

$\forall \varepsilon > 0, \exists N(\varepsilon)$ s.t. $|S(z) - S_n(z)| < \varepsilon, \ \forall n > N(\varepsilon)$

*A uniformly convergent series can be integrated/differentiated term-by-term (only sufficient).*
**Thm 1** Taylor expansion (finite terms)

If \( f(z) \) is analytic throughout the region bounded by a simple closed curve \( C \) and both \( z \) and \( a \) are interior to \( C \), then

\[
f(z) = f(a) + f'(a)(z-a) + f''(a)\frac{(z-a)^2}{2!} + \cdots + f^{(n-1)}(a)\frac{(z-a)^{n-1}}{(n-1)!} + R_n
\]

where
\[
R_n = \frac{(z-a)^n}{2\pi i} \oint_C \frac{f(t)dt}{(t-a)^n(t-z)}
\]

pf:

From Cauchy's integral theorem,

\[
f(z) = \frac{1}{2\pi i} \oint_C \frac{f(t)dt}{t-z} = \frac{1}{2\pi i} \oint_C \frac{f(t)}{t-a} \frac{1}{1-(z-a)/(t-a)} dt
\]
Applying the identity
\[ \frac{1}{1-u} = 1 + u + u^2 + \cdots + u^{n-1} + \frac{u^n}{1-u} \] (long division)
to the factor \[ \frac{1}{1-(z-a)/(t-a)} \], we have
\[ f(z) = \frac{1}{2\pi i} \oint \frac{f(t)}{t-a} \left[ 1 + \frac{z-a}{t-a} + \frac{(z-a)^2}{t-a} + \cdots + \frac{(z-a)^{n-1}}{t-a} + \frac{(z-a)^n}{1-(z-a)/(t-a)} \right] dt \]
\[ = \frac{1}{2\pi i} \oint \frac{f(t)}{t-a} dt + \frac{z-a}{2\pi i} \oint \frac{f(t)}{(t-a)^2} dt + \cdots + \frac{(z-a)^{n-1}}{2\pi i} \oint \frac{f(t)dt}{(t-a)^n} + \frac{(z-a)^n}{2\pi i} \oint \frac{f(t)dt}{(t-a)^n(t-z)} \]

Recall the generalized Cauchy integral formula
\[ f^{(n)}(a) = \frac{n!}{2\pi i} \oint \frac{f(t)dt}{(t-a)^{n+1}} \]
Hence,
\[ f(z) = f(a) + f'(a)(z-a) + \cdots + f^{(n-1)}(a) \frac{(z-a)^{n-1}}{(n-1)!} + \frac{(z-a)^n}{2\pi i} \oint \frac{f(t)dt}{(t-a)^n(t-z)} \]
Q.E.D.
It can be shown below that the residual term $R_n$ in the Taylor expansion of Thm1 approaches zero as $n \to \infty$.

**Thm 2  Taylor series**

$$f(z) = f(a) + f'(a)(z-a) + f''(a)\frac{(z-a)^2}{2!} + \cdots$$

is a valid representation of $f(z)$ at all points in the interior of any circle having its center at $z = a$ and within which $f(z)$ is analytic.

**Note:** The existence of Taylor series in the neighborhood of a point can be regarded as the condition for the function $f(z)$ to be analytic at that point.
pf:
Knowing \(|t - a| = r_2\), \(|z - a| < r_1\), \(|t - z| > r_2 - r_1\), \(|f(t)| \leq M\)
where \(M = \max |f(z)|\) on \(C_2\), we have

\[
|R_n(z)| = \left| \frac{(z-a)^n}{2\pi i} \oint_{C_2} \frac{f(t)dt}{(t-a)^n(t-z)} \right| \leq \left| \frac{(z-a)^n}{2\pi i} \oint_{C_2} \frac{|f(t)|dt}{|t-a|^n|t-z|} \right|
\]

\[
< \frac{r_1^n}{2\pi} \oint_{C_2} \frac{M|dt|}{r_2^n(r_2-r_1)} = \frac{r_1^nM}{2\pi r_2^n(r_2-r_1)} 2\pi r_2 = M(r_1/r_2)^n \frac{r_2}{r_2-r_1}
\]

\[
\therefore 0 < r_1 < r_2 \therefore \lim_{n \to \infty} (r_1/r_2)^n = 0 \Rightarrow \lim_{n \to \infty} |R_n(z)| = 0 \quad \text{Q.E.D.}
\]
Thm 3

If \( a_0 + a_1(z - a) + a_2(z - a)^2 + \cdots \) converges for \( z = z_1 \), then it converges absolutely for \( \forall z \ni |z - a| < |z_1 - a| \) - (interior of the circle \( |z - a| = |z_1 - a| \))
pf:

\[ \sum_{n=0}^{\infty} a_n (z_1 - a)^n \text{ converges} \]

\[ \therefore \lim_{n \to \infty} a_n (z_1 - a)^n = 0 \Rightarrow \exists N \ni |a_n (z_1 - a)^n| < 1 \text{ for } n \geq N \]

For \( |z - a| < |z_1 - a| \),

\[ |a_n (z - a)^n| = |a_n (z_1 - a)^n| \cdot \left| \frac{z - a}{z_1 - a} \right|^n < 1 \cdot \left| \frac{z - a}{z_1 - a} \right|^n \text{ for } n \geq N \]

Knowing the geometric series \( \sum_{n=1}^{\infty} \left| \frac{z - a}{z_1 - a} \right|^n \) converges for \( |z - a| < |z_1 - a| \).

By comparison test, \( \sum_{n=0}^{\infty} |a_n (z - a)^n| \) converges absolutely.

* If \( a = 0 \), the series is called Maclaurin's series.
Thm 4
If is impossible for the Taylor series of \( f(z) \) to converge to \( f(z) \) outside the circle \( C \) whose center is \( z = a \) and whose radius is the distance from \( a \) to the nearest singularity.

\( \times \): singularity

\( ROC \): radius of convergence
Thm 4 Binomial expansion

\[(s + t)^n = s^n + ns^{n-1}t + \frac{n(n-1)}{2!} s^{n-2}t^2 + \frac{n(n-1)(n-2)}{3!} s^{n-3}t^3 + \ldots \]

\[\forall n \text{ if } |t| < |s|\]

Thm 5

If \( f(z) \) can be represented in the neighborhood of \( z = a \) by a series of the form \( \sum_{n=1}^{\infty} a_n (z - a)^n \), the representation is unique.
Ex. Expand \( f(z) = \frac{3}{3z - z^2} \) about \( z = 1 \)

Sol: \( f(z) = \frac{1}{z} + \frac{1}{3 - z} = [1 + (z - 1)]^{-1} + [2 - (z - 1)]^{-1} \)

Using binomial expansion,

\[
(s + t)^n = s^n + ns^{n-1}t + \frac{n(n-1)}{2!} s^{n-2}t^2 + \frac{n(n-1)(n-2)}{3!} s^{n-3}t^3 + \cdots \quad \forall |t| < |s|
\]

\[
\Rightarrow f(z) = 1 - (z - 1) + (z - 1)^2 - \cdots + (-1)^n (z - 1)^n + \cdots \quad (\forall |z - 1| < 1)
\]

\[
+ 2^{-1} + 2^{-2} (z - 1) + \cdots + 2^{-(n-1)} (z - 1)^n \cdots \quad (\forall |z - 1| < 2)
\]

\[
= \frac{3}{2} - \frac{3}{4} (z - 1) + \cdots + \left[ \frac{1}{2^{n-1}} + (-1)(z - 1)^n \right] \quad (\forall |z - 1| < 1)
\]

\[ \text{ROC : } ROC_1 \cap ROC_2 \]

The nearest singularity

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Mingsian R. Bai

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Thm 7  Liouville theorem
If $|f(z)| < \infty$ and analytic $\forall z$ (entire complex plane),
then $f(z) = \text{constant}$.

pf:

$$f(z) = f(0) + f'(0)z + \cdots + \frac{f^{(n)}(0)}{n!} z^n + \cdots$$

If $|z| \to \infty$, then $|f(z)| \to \infty$ unless $f(z) = f(0)$

Q.E.D.
Laurent’s Expansion

Taylor's expansion:  

Sometimes we need:

Note:
- Including non-negative and negative powers
- Important in $z$-transform and discrete signal and system analysis
**Thm 1  Laurent expansion/series**

If \( f(z) \) is **analytic** in an annulus \( R \) bounded by two concentric circles with center \( a \), then at any point in \( R \),

\[
f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n
\]

\[
= \cdots + \frac{a_{-2}}{(z-a)^2} + \frac{a_{-1}}{(z-a)} + a_0
\]

\[
+ a_1(z-a) + a_2(z-a)^2 + \cdots,
\]

where \( a_n = \frac{1}{2\pi i} \oint_C \frac{f(t)dt}{(t-a)^{n+1}} \) taken around any simple closed path \( C \) that lies in an open annulus.

---

*Mingsian R. Bai*
pf: Cauchy's integral theorem

\[
f(z) = \frac{1}{2\pi i} \left( \oint_{C_2} f(t) \frac{dt}{t-z} + \oint_{C_1} f(t) \frac{dt}{t-z} \right) = \frac{1}{2\pi i} \oint_{C_2} f(t) \frac{dt}{t-z} - \frac{1}{2\pi i} \oint_{C_1} f(t) \frac{dt}{t-z}
\]

\[
= \frac{1}{2\pi i} \oint_{C_2} \frac{f(t) \, dt}{t-a} \frac{1}{1 - \left(\frac{z-a}{t-a}\right)} + \frac{1}{2\pi i} \oint_{C_1} \frac{f(t) \, dt}{z-a} \frac{1}{1 - \left(\frac{t-a}{z-a}\right)}
\]

Apply the identity

\[
\frac{1}{1-u} = 1 + u + u^2 + \cdots + \frac{u^n}{1-u} \quad \text{(long division)}
\]

\[
f(z) = \frac{1}{2\pi i} \oint_{C_2} \frac{f(t) \left[1 + \frac{z-a}{t-a} + \cdots + \left(\frac{z-a}{t-a}\right)^{n-1} + \frac{(z-a)^n / (t-a)^n}{1-(z-a)/(t-a)}\right] \, dt}{1 - \left(\frac{z-a}{t-a}\right)} + \frac{1}{2\pi i} \oint_{C_1} \frac{f(t) \left[1 + \frac{t-a}{z-a} + \cdots + \left(\frac{t-a}{z-a}\right)^{n-1} + \frac{(t-a)^n / (z-a)^n}{1-(t-a)/(z-a)}\right] \, dt}{1 - \left(\frac{t-a}{z-a}\right)}
\]

Note that \(\left|\frac{z-a}{t-a}\right| < 1\) in \(C_2\) and \(\left|\frac{t-a}{z-a}\right| < 1\) in \(C_1\), which is essential for convergence.

ROC: interior of \(C_2 \cap\) exterior of \(C_1 =\) open annulus bounded by \(C_1\) and \(C_2\).
Positive powers of \((z-a)\)

\[
f(z) = \frac{1}{2\pi i} \oint_{C_2} \frac{f(t)dt}{t-a} + \frac{z-a}{2\pi i} \oint_{C_2} \frac{f(t)dt}{(t-a)^2} + \cdots + \frac{(z-a)^{n-1}}{2\pi i} \oint_{C_2} \frac{f(t)dt}{(t-a)^n} + R_{n2}
\]

\[
+ \frac{1}{2\pi i(z-a)} \oint_{C_1} f(t)dt + \frac{1}{2\pi i(z-a)^2} \oint_{C_1} (t-a)f(t)dt
\]

\[
+ \cdots + \frac{1}{2\pi i(z-a)^n} \oint_{C_1} (t-a)^{n-1}f(t)dt + R_{n1} = \sum_{n=-\infty}^{\infty} a_n(z-a)^n,
\]

Negative powers of \((z-a)\)

where \(C_1\) and \(C_2\) have been replaced by any closed curve \(C\) in the annulus bounded by \(C_1\) and \(C_2\) in the integrals for \(a_n\) (principle of path deformation, page 30),

\[
a_n = \frac{1}{2\pi i} \oint_{C} \frac{f(t)dt}{(t-a)^{n+1}}
\]

\[
R_{n2} = \frac{(z-a)^n}{2\pi i} \oint_{C_2} \frac{f(t)dt}{(t-a)^n(t-z)} ; R_{n1} = \frac{1}{2\pi i(z-a)^n} \oint_{C_1} \frac{(t-a)^n f(t)dt}{z-t}
\]
It has also been proved in Taylor's expansion, \(|R_{n2}| \to 0\) as \(n \to \infty\)

On the other hand, to prove \(|R_{n1}| \to 0\) as \(n \to \infty\), we assume

\[ |t - a| = r_1, \quad |z - a| = \rho > r_1, \quad |f(t)| \leq M = \max |f(z)| \text{ on } C_1 \]

\[ |z - t| = |(z - a) - (t - a)| \geq \rho - r_1 \text{ (triangle inequality)}. \]

Then,

\[
|R_{n1}| = \left| \frac{1}{2\pi i(z - a)^n} \oint_{C_1} \frac{(t - a)^n f(t)dt}{z - t} \right| \leq \frac{1}{2\pi |z - a|^n} \oint_{C_1} \frac{t - a^n |f(t)| dt}{|z - t|} \\
\leq \frac{r_1^n M}{2\pi \rho^n (\rho - r_1)} \oint_{C_1} |dt| = \frac{M}{2\pi} \left( \frac{r_1}{\rho} \right)^n \frac{2\pi r_1}{\rho - r_1} \to 0 \text{ as } n \to \infty
\]

Q.E.D.
Note:

(1) In the above proof, the coefficient of the positive powers in Laurent's expansion $a_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(t)dt}{(t-a)^{n+1}}$ cannot be replaced by $\frac{f^{(n)}(a)}{n!}$ like Taylor expansion since $f(z)$ is not analytic throughout the entire interior of $C_2$. Here, Cauchy's integral formula does not apply.

(2) **Uniqueness**: If an expansion of Laurent form can be found by any means, it must be the Laurent expansion.

Ex. Binomial expansion, Geometric series, etc.

$\Rightarrow$ The **uniqueness** is essential for one to avoid evaluating the integrals in finding $a_n$, but to use simpler alternatives.
(3) The open annulus can be enlarged by decreasing $C_1$ and increasing $C_2$ until each of the two circles reaches a singularity, where $f(z)$ ceases to be analytic.

(4) Important special case: $a$ is the only singular point inside $C_1$, giving the convergence in a disk except at the center ($0 < |z - a| < \varepsilon$, deleted neighborhood). In this case, the negative powers of the Laurent series of $f(z)$ is called the principal part of the singularity of $f(z)$ at $a$. In particular, the coefficient of the term ($n = -1$)

$$a_{-1} = \frac{1}{2\pi i} \oint_{C} \frac{f(t)dt}{(t-a)^{-1+1}} = \frac{1}{2\pi i} \oint_{C} f(t)dt = \text{Res}(a)$$

$$\Rightarrow \oint_{C} f(z)dz = 2\pi i \cdot \text{Res}(a)$$

prompts the development of the residue integration technique.

cf. $\oint_{C} f(z)dz = 0$ if $f(z)$ is analytic throughout the interior of $C$. 
Ex.

Find the Laurent expansion of \( f(z) = \frac{7z-2}{(z+1)z(z-2)} \) in \( 1 < |z+1| < 3 \)

Sol: \( a = -1 \)

\[
f(z) = \frac{-3}{z+1} + \frac{1}{z} + \frac{2}{z-2} = \frac{-3}{z+1} + \frac{1}{(z+1)-1} + \frac{2}{(z+1)-3}
\]

(PFE) \hspace{1cm} (1) \hspace{1cm} (2) \hspace{1cm} (3)

Geometric series: \( \frac{1}{1-u} = 1 + u + u^2 + \cdots + u^n + \cdots, \quad |u| < 1 \)

(1): \( \frac{-3}{z+1} \) for \( |z+1| > 0 \) \hspace{1cm} 觀察法

(2): \[
\frac{1}{(z+1)-1} = \frac{-1}{1-(z+1)} = -[1+(z+1)+(z+1)^2 + \cdots], \quad |z+1| < 1 \quad (\times)
\]
or

\[
\frac{1}{(z+1)-1} = \frac{(z+1)^{-1}}{1-(z+1)^{-1}} = (z+1)^{-1}[1+(z+1)^{-1}+(z+1)^{-2}+\cdots], \quad |(z+1)^{-1}| < 1
\]

\[
= (z+1)^{-1} + (z+1)^{-2} + (z+1)^{-3} + \cdots, \quad 1 < |z+1| \quad (\bigcirc)
\]

(3): \[
\frac{2}{(z+1)-3} = \frac{-2/3}{1-3^{-1}(z+1)} = (-\frac{2}{3})[1+3^{-1}(z+1)+3^{-2}(z+1)^2+\cdots]
\]

for \(|3^{-1}(z+1)| < 1 \text{ or } |z+1| < 3 \quad (\bigcirc) \quad \text{(interior of circle)}
\]

or \[
\frac{2}{(z+1)-3} = \frac{2(z+1)^{-1}}{1-3(z+1)^{-1}} = 2(z+1)^{-1}[1+3(z+1)^{-1}+3^2(z+1)^{-2}+\cdots]
\]

for \(|3(z+1)^{-1}| < 1 \text{ or } 3 < |z+1| \quad (\times) \quad \text{(exterior of circle)}
Thus,

\[ f(z) = -3(z + 1)^{-1} + [(z + 1)^{-1} + (z + 1)^{-2} \cdots] \]

\[ + \left[ -\frac{2}{3} - \frac{2}{9} (z + 1) - \frac{2}{27} (z + 1)^2 - \cdots \right] \]

\[ = \cdots + (z + 1)^{-3} + (z + 1)^{-2} - 2(z + 1)^{-1} \]

\[ - \frac{2}{3} - \frac{2}{9} (z + 1) - \frac{2}{27} (z + 1)^2 - \frac{2}{81} (z + 1)^3 - \cdots \]

Overall, \( ROC = |z + 1| > 0 \cap |z + 1| > 1 \cap 0 < |z + 1| < 3 = 1 < |z + 1| < 3 \)
Note:

(1) \( f(z) \) has two other expansions for \( 0 < |z+1| < 1 \) and \( |z+1| > 3 \)

(2) **Positive powers** → ROC is the **interior of a circle**

Ex. \( \frac{1}{1-z} = 1 + z + z^2 + \cdots \) for \(|z| < 1\)

(3) **Negative powers** → ROC is the **exterior of a circle**

Ex. \( \frac{1}{1-z} = -\frac{1}{z-1} = -\frac{-z^{-1}}{1-z^{-1}} = -z^{-1}[1 + z^{-1} + z^{-2} \cdots] \)

\[ = -z^{-1} - z^{-2} - z^{-3} - \cdots \]

for \( |z^{-1}| < 1 \) or \( 1 < |z| \)
III. The Theory of Residues

Singularities

(1) $f(z)$ is not analytic and not differentiable ($\frac{df(z)}{dz}$ does not exist).

(2) branch points and branch cuts of multi-valued functions

Def. isolated singular point

$f(z)$ is analytic in the immediate neighborhood of $z = a$,
i.e., $0 < |z - a| < \varepsilon$ except at $z = a$

In this case, the Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - a)^n, \quad 0 < |z - a| < \varepsilon$$

is valid.

Note: A branch point is not an isolated singular point.
(1) **Def.** removable singularity

$f(z)$ would become analytic at $z = a$ if it is appropriately redefined.

**Ex:** \[ \sin z = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \]

\[ \frac{\sin z}{z} \text{ can be redefined as } \begin{cases} 
1, & z = 0 \\
\frac{\sin z}{z}, & z \neq 0 
\end{cases} \]

(2) **Def.** pole

If the Laurent expansion of $f(z)$ in the deleted neighborhood of an isolated singular point ($\lim f(z) \to \infty$ as $z \to a$) contains only a finite number of negative powers of $(z - a)$, then $z = a$ is called a pole of $f(z)$.
If \((z-a)^{-m}\), \(m\) being a finite integer, is the numerically highest negative power in the Laurent expansion, the pole is said to be of order \(m\) at \(z = a\), i.e., \((z-a)^m f(z)\) is analytic at \(z = a\).

The sum of all the terms containing negative powers

\[
\frac{a_{-m}}{(z-a)^m} + \cdots + \frac{a_{-1}}{(z-a)}
\]

is called the principal part of \(f(z)\) at \(z = a\).

**Ex.**

\[
f(z) = \frac{1}{z(z-1)^2}
\]

\[
(z-1)^2 f(z) = \frac{1}{z} = \frac{1}{1+(z-1)} = 1 + (z-1) - (z-1)^2 + \cdots
\]

is analytic for \(|z-1| < 1\) (deleted neighborhood at \(z = 1\))
It follows that \( f(z) \) has a pole of order 2 at \( z = 1 \).

More precisely,

\[
f(z) = \frac{1}{z(z-1)^2} = [1+(z-1)]^{-1} = \frac{1-(z-1)+(z-1)^2 - \cdots}{(z-1)^2}
\]

\[
= \frac{1}{(z-1)^2} - \frac{1}{(z-1)} + 1 - (z-1) + \cdots \quad 0 < |z-1| < 1
\]

Thus, \( f(z) \) has a pole of order 2 at \( z = 1 \) and its principal part is

\[
\frac{1}{(z-1)^2} - \frac{1}{(z-1)}
\]
However, we can also write

\[ f(z) = \frac{1}{z(z-1)^2} = \frac{[(z-1)+1]^{-1}}{(z-1)^2} = \frac{(z-1)^{-1}[1+(z-1)^{-1}]^{-1}}{(z-1)^2} \]

\[ = \frac{(z-1)^{-1} - (z-1)^{-2} + (z-1)^{-3} \cdots}{(z-1)^2} \]

\[ = \cdots + \frac{1}{(z-1)^5} - \frac{1}{(z-1)^4} + \frac{1}{(z-1)^3} \quad |z-1|>1 \]

The fact that this expansion contains infinitely many negative powers does not contradict the preceding conclusion (pole of order 2 at \( z = 1 \)). For this series is valid only outside the circle \(|z-1|=1\), whereas, by definition, the presence of poles is determined by the Laurent expansion which is valid in the innermost annulus, or deleted neighborhood of the singularity in question.
Thm

$f(z)$ has a pole of order $m$ at $z = a \iff f(z)$ has a Laurent expansion in the neighborhood of $z = a$ with $a_n = 0$ when $n < -m$ and $a_{-m} \neq 0$

Def. essential singular point

If the Laurent expansion of $f(z)$ in the neighborhood of an isolated singular point $z = a$ contains infinitely many negative powers of $(z - a)$, then $z = a$ is called an essential singular point.

Ex

$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$

has an essential singular point at $z = 0$
Thm 1  Residue Theorem

If \( f(z) \) is analytic within and on a simple closed curve \( C \), except at a finite number of isolated singular points \( z_1, \ldots, z_N \) lying inside \( C \), then

\[
\oint_C f(z) \, dz = 2\pi i \sum_{k=1}^{N} \text{Res} f(z_k)
\]

where \( \text{Res} f(z_k) \) denotes the residues of \( f(z) \) at its singular points within \( C \), or the coefficient \( a_{-1} \) of the term \( (z - z_k)^{-1} \) in the Laurent expansion of \( f(z) \) about \( z_k \).

\[
a_{-1}(z_k) = \frac{1}{2\pi i} \oint_{C_k} f(z) \, dz
\]
Pf:
Apply Cauchy's theorem to the multiply connected region in which \( f(z) \) is everywhere analytic

\[
\frac{1}{2\pi i} \oint_C f(z)\,dz + \frac{1}{2\pi i} \oint_{C_1} f(z)\,dz + \cdots + \frac{1}{2\pi i} \oint_{C_N} f(z)\,dz = 0
\]

\[
\frac{1}{2\pi i} \oint_C f(z)\,dz = \frac{1}{2\pi i} \oint_{C_1} f(z)\,dz + \cdots + \frac{1}{2\pi i} \oint_{C_N} f(z)\,dz
\]

\[
= (a_{-1})_{z_1} + \cdots + (a_{-1})_{z_N}
\]

\[
= \sum_{k=1}^{N} \text{Res} \, f(z_k) \quad (\because \text{Laurent series, p.68})
\]

\[
\Rightarrow \oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{N} \text{Res} \, f(z_k)
\]

Q.E.D.
Thm2

If \( f(z) \) has a pole of order \( m \) at \( z = a \), then the residue of \( f(z) \) at \( z = a \) is

\[
a_{-1} = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left[ (z-a)^m f(z) \right]
\]

pf:

\[
(z-a)^m f(z) = a_{-m} + a_{-m+1}(z-a) + \cdots + a_{-1}(z-a)^{m-1} + \cdots
\]

\[
\frac{d^{m-1}}{dz^{m-1}} \left[ (z-a)^m f(z) \right] = (m-1)! \left[ a_{-1} + a_0(z-a) + \cdots \right]
\]

\[
\Rightarrow \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left[ (z-a)^m f(z) \right] = (m-1)! a_{-1} \quad \text{Q.E.D.}
\]
Special case

\[ f(z) = \frac{N(z)}{D(z)}. \] If \( m = 1 \) (simple pole), then

\[ \text{Res} \ f(a) = \lim_{z \to a} (z-a) f(z) = \frac{N(a)}{D'(a)} \]

If \( D(z) \) is a polynomial, \( D'(z) \) can be simply obtained by deleting \( (z-a) \) in the factored \( D(z) \).

**Pf:**

Let \( D(z) = (z-a)Q(z) \).

\[ \lim_{z \to a} (z-a) f(z) = \lim_{z \to a} \frac{N(z)}{Q(z)} = \frac{N(a)}{Q(a)} \]

\[ D'(z) = \frac{d}{dz} [(z-a)Q(z)] = Q(z) + (z-a)Q'(z) \]

\[ \frac{N(a)}{D'(a)} = \frac{N(a)}{Q(a)} = \lim_{z \to a} (z-a) f(z) \quad \text{Q.E.D.} \]
Ex. \[ \oint_C \frac{-3z + 4}{z(z-1)(z-2)} \, dz = ? \quad C : |z| = \frac{3}{2} \]

Sol:

\[ f(z) = \frac{-3z + 4}{z(z-1)(z-2)} = \frac{-3z + 4}{z^3 - 3z^2 + 2z} \]

Simple poles: \( z = 0, 1, 2 \)

However, only \( z = 0, 1 \) are inside \( C : |z| = \frac{3}{2} \)

\[ \oint_C f(z) \, dz = 2\pi i \left[ \text{Res } f(0) + \text{Res } f(1) \right] \]

\[ = 2\pi i \left[ \frac{-3z + 4}{3z^2 - 6z + 2} \bigg|_{z=0} + \frac{-3z + 4}{3z^2 - 6z + 2} \bigg|_{z=1} \right] \]

\[ = 2\pi i \left( \frac{4}{2} + \frac{1}{-1} \right) = 2\pi i(1) = 2\pi i \]
Evaluation of Real Definite Integrals

\[ I = \int_{0}^{2\pi} R(\sin \theta, \cos \theta) d\theta \]

Let \( z = e^{i\theta}, \quad dz = ie^{i\theta} d\theta = iz d\theta \)

\[ \Rightarrow d\theta = \frac{dz}{iz}, \quad |z| = 1 \text{ (unit circle)} \]

\[ \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i} , \]

\[ \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2} \]

\[ I = \int_{0}^{2\pi} R(\sin \theta, \cos \theta) d\theta = \oint_{|z|=1} F(z) \frac{dz}{iz} \]

(contour integral evaluated on the unit circle)
Ex \( I_1 = \int_0^{2\pi} \frac{d\theta}{A + B\cos\theta} \) \((A^2 > B^2, \ A > 0)\)

Sol: \( I_1 = \oint_{|z|=1} \frac{dz / iz}{A + \frac{B(z + z^{-1})}{2}} = \oint_{|z|=1} \frac{-2idz}{Bz^2 + 2Az + B} \)

Poles: \( z = \frac{-2A \pm \sqrt{4A^2 - 4B^2}}{2B} = -\frac{A}{B} \pm \frac{\sqrt{A^2 - B^2}}{B} \)

Note that \( z_1z_2 = 1, \ z_1 = -\frac{A}{B} + \frac{\sqrt{A^2 - B^2}}{B} \) is inside \( |z| = 1 \).

\[ \text{Res } f(z) = \left. \frac{-2i}{2Bz + 2A} \right|_{z = z_1} = \frac{-2i}{-2A + 2\sqrt{A^2 - B^2} + 2A} = \frac{-2i}{2\sqrt{A^2 - B^2}} = \frac{2\pi}{\sqrt{A^2 - B^2}} \]

\[ I_1 = 2\pi i \text{ Res } f(z) = 2\pi i \frac{-2i}{2\sqrt{A^2 - B^2}} = \frac{2\pi}{\sqrt{A^2 - B^2}} \]
Evaluation of Real Definite Integrals

ML-rule:

\[
\left| \int_{C} f(z) \, dz \right| = \left| \sum_{k=0}^{N} f(z_k) \Delta z_k \right| \leq \sum_{k=0}^{N} \left| f(z_k) \Delta z_k \right| \leq \sum_{k=0}^{N} \left| f(z_k) \right| \Delta z_k
\]

\[N \to \infty \Rightarrow\]

\[
\left| \int_{C} f(z) \, dz \right| \leq \int_{C} \left| f(z) \right| \, dz = \int_{C} \left| f(z) \right| \, ds
\]

\[
\leq \int_{C} M ds \quad (M = \max \left| f(z) \right| \text{ on the path } C)
\]

\[
= M \int_{C} ds = ML
\]

Thus, \[
\left| \int_{C} f(z) \, dz \right| \leq ML \quad \text{(ML-rule)}
\]
Type \( I = \int_{-\infty}^{\infty} f(x)dx \) (improper integral)

where \( f(x) \) is a rational function, \( f(x) = \frac{p(x)}{q(x)} \),
\( \deg q(x) - \deg p(x) \geq 2 \) and \( q(x) = 0 \) has no real roots.

Ex \( I = \int_{-\infty}^{\infty} \frac{x^2}{1+x^4} \ dx = ? \)

Sol: \( f(z) = \frac{z^2}{1+z^4} \)

\[
\int_{-R}^{R} \frac{x^2}{1+x^4} \ dx + \int_{C_{R}} f(z) \ dz = 2\pi i \sum_{k} \text{Res} \ f(a_k), \ R \to \infty
\]

On \( C_{R}, \ |f(z)| = \left| \frac{z^2}{1+z^4} \right| \leq \frac{|z|^2}{|z|^4 - 1} = \frac{R^2}{R^4 - 1} = M \ (R > 1) \)
By the ML-rule, \[
\left| \int_{C_R} f(z)\,dz \right| \leq ML = \frac{R^2}{R^4 - 1} \pi R = \frac{\pi R^3}{R^4 - 1} \to 0 \text{ as } R \to \infty.
\]

Thus, \[
I = \lim_{R \to \infty} \int_{-R}^{R} \frac{x^2}{1 + x^4} \,dx + 0 = 2\pi i \sum_{k} \text{Res } f(a_k)
\]

But only the poles inside the closed contour are \(a_1 = e^{i\pi/4}, a_2 = e^{i3\pi/4}\)

\[
\text{Res } f(a_k) = \frac{z^2}{4z^3} \bigg|_{z=a_k} = \frac{1}{4a_k} \quad k = 1, 2
\]

Thus, \[
I = \int_{-\infty}^{\infty} \frac{x^2}{1 + x^4} \,dx = 2\pi i \left( \frac{1}{4} e^{-i\pi/4} + \frac{1}{4} e^{-i3\pi/4} \right) = \frac{\pi}{\sqrt{2}} \quad #
\]

* In fact, if \(\deg q(x) - \deg p(x) = 2\) and \(q(x) = 0\) has no real roots

\[
M \sim \frac{1}{|z|^2} = \frac{1}{R^2}, \quad L = \pi R \Rightarrow \left| \int_{C_R} f(z)\,dz \right| \leq ML = \frac{\pi}{R} \to 0 \text{ as } R \to \infty
\]
More precisely,

$$|f(z)| = \left| \frac{p(z)}{q(z)} \right| = \left| \frac{a_n z^n + a_{n-1} z^{n-1} + \cdots}{b_{n+2} z^{n+2} + b_{n+1} z^{n+1} + \cdots} \right|$$

$$\leq \frac{|a_n z^n| + |a_{n-1} z^{n-1}| + \cdots}{|z-c_1| (|z-c_2|) \cdots (|z-c_{n+2}|)} \leq \frac{a_n R^n + a_{n-1} R^{n-1} + \cdots}{(|z| - |c_1|) (|z| - |c_2|) \cdots}$$

$$= \frac{a_n R^n + a_{n-1} R^{n-1} + \cdots}{(R - |c_1|) (R - |c_2|) \cdots (R - |c_{n+2}|)} \sim \frac{K}{R^2} \to 0 \quad \text{as} \quad R \to \infty$$
Type Fourier transform

\[ I = \int_{-\infty}^{\infty} f(x) \cos mx \, dx \quad \text{or} \quad \int_{-\infty}^{\infty} f(x) \sin mx \, dx, \quad m \geq 0 \]

are the real and imaginary parts of \( \int_{-\infty}^{\infty} e^{imx} f(x) \, dx \)

Thm If \( f(x) \) is a rational function \( \deg q(x) - \deg p(x) \geq 2 \), then

\[ \int_{-\infty}^{\infty} e^{imx} f(x) \, dx = 2\pi i \sum_{k} \text{Res} \{ e^{imz} f(z) \} \quad (m \geq 0) \]

where \( a_k \) are the poles of \( f(z) \) in the upper-half plane.

Pf: \( |e^{imz}| = |e^{imRe^{i\theta}}| = |e^{-imR(\cos \theta + i\sin \theta)}| = e^{-mR\sin \theta} \)

\[ |e^{-imz}| = |e^{-imRe^{i\theta}}| = |e^{-imR(\cos \theta + i\sin \theta)}| = e^{mR\sin \theta} \quad (m \geq 0) \]

\( \Rightarrow |e^{imz}| \rightarrow 0 \) on \( C_R \) as \( R \rightarrow \infty \) in the upper-half-plane \( (\sin \theta > 0) \)
\[ |e^{im\theta} f(z)| = e^{-mR\sin \theta} |f(z)| \]

\[ \leq |f(z)| \leq \frac{K}{R^2} = M \quad (\because \deg q(x) - \deg p(x) \geq 2) \]

Thus,
\[ \left| \int_{C_R} e^{im\theta} f(z) \, dz \right| \leq ML = \frac{K}{R^2} \pi R = \pi \frac{K}{R} \rightarrow 0 \quad \text{as} \quad R \rightarrow \infty \]

\[ \lim_{R \rightarrow \infty} \left\{ \int_{-R}^{R} e^{imx} f(x) \, dx + \int_{C_R} e^{im\theta} f(z) \, dz \right\} = 2\pi i \sum_k \text{Res} \ f(a_k) \]

Q.E.D.
\[ I = \int_{-\infty}^{\infty} \frac{e^{imx} \, dx}{a^2 + x^2}, \quad m \geq 0, \quad a > 0 \]

Sol: Poles: \( z = \pm ia \)

\[ \int_{-\infty}^{\infty} \frac{e^{imx} \, dx}{a^2 + x^2} = 2\pi i \cdot \text{Res}_{z=ia} \left\{ \frac{e^{imz}}{a^2 + z^2} \right\} = 2\pi i \left( \frac{e^{imz}}{z + ia} \right)_{z=ia} \]

\[ = 2\pi i \frac{e^{-ma}}{2ia} = \frac{\pi}{a} e^{-ma} \quad m \geq 0, \quad a > 0 \]

Thus, \( \int_{-\infty}^{\infty} \frac{\cos mx}{a^2 + x^2} \, dx = \frac{\pi}{a} e^{-ma} \) (even function) \( \Rightarrow \int_{0}^{\infty} \frac{\cos mx}{a^2 + x^2} \, dx = \frac{\pi}{2a} e^{-ma} \)

\[ \int_{-\infty}^{\infty} \frac{\sin mx}{a^2 + x^2} \, dx = 0 \] (odd function)

* How about \( I = \int_{-\infty}^{\infty} f(x) e^{imx} \, dx \) (\( m < 0 \)) ?

(1) Lower-half plane, \( I = -2\pi i \sum_{k} \text{Res}_{z=d_k} \left\{ e^{imz} f(z) \right\} \)

(minus sign is due to clockwise integration)

or (2) Replacing \( x \) by \( -x \)

\[ |e^{imz}| = e^{-mR \sin \theta} \rightarrow 0 \text{ as } R \rightarrow \infty, \sin \theta < 0 \]
Theorems on Limiting Contours

Thm 1  If on a circular arc $C_R$ (with span angle $\alpha$) with radius $R$ and center at $z = 0$, $zf(z) \to 0$ uniformly as $R \to \infty$ (independent of angle), then

$$\lim_{R \to \infty} \int_{C_R} f(z) \, dz = 0$$

Pf: $\left|zf(z)\right| \leq K_R \to 0$ as $R \to \infty \Rightarrow \left|f(z)\right| \leq \frac{K_R}{R} = M$

By the ML-rule,

$$\left|\int_{C_R} f(z) \, dz\right| \leq \frac{K_R}{R} \cdot \alpha R \to 0 \text{ as } R \to \infty$$

Q.E.D.
**Thm 2** Jordan’s lemma (less restrictive than the previous theorem)

$C_R$ is a circular arc with radius $R$ and center at $z = 0$, intercepting an angle $\alpha = \theta_1 - \theta_0$. If on $C_R$, $f(z) \to 0$ uniformly as $R \to \infty$, then

1. $\lim_{R \to \infty} \int_{C_R} e^{imz} f(z) \, dz = 0$ $(m > 0)$

2. $\lim_{R \to \infty} \int_{C_R} e^{-imz} f(z) \, dz = 0$ $(m > 0)$

3. $\lim_{R \to \infty} \int_{C_R} e^{mz} f(z) \, dz = 0$ $(m > 0)$

4. $\lim_{R \to \infty} \int_{C_R} e^{-mz} f(z) \, dz = 0$ $(m > 0)$
Pf: \( I_R = \int_{C_R} e^{imz} f(z) dz, \quad m > 0 \)

Let \( z = Re^{i\theta}, \ dz = Rie^{i\theta} d\theta, \ |dz| = Rd\theta, \ |f(z)| \leq K_R \to 0 \) as \( R \to \infty \)

\[
|I_R| = \left| \int_{C_R} e^{imz} f(z) dz \right| \leq \int_{C_R} \left| e^{imR(\cos\theta+i\sin\theta)} \right| |f(z)| |dz| = \int_{\theta_0}^{\theta_1} e^{-mR\sin\theta} K_R Rd\theta
\]

\[
= RK_R \int_{\theta_0}^{\theta_1} e^{-mR\sin\theta} d\theta \leq RK_R \int_{0}^{\pi} e^{-mR\sin\theta} d\theta = 2RK_R \int_{0}^{\pi/2} e^{-mR\sin\theta} d\theta
\]

Knowing \( \sin\theta \geq \frac{2\theta}{\pi} \) for \( 0 \leq \theta \leq \frac{\pi}{2} \) (Jordan's inequality)

\[
|I_R| \leq 2RK_R \int_{0}^{\pi/2} e^{-mR\frac{2\theta}{\pi}} d\theta = \frac{\pi}{m} K_R \left(1-e^{-mR}\right) < \frac{\pi}{m} K_R \to 0 \text{ as } R \to \infty
\]

\[
\Rightarrow \lim_{R \to \infty} |I_R| = 0 \Rightarrow \lim_{R \to \infty} I_R = 0
\]

Q.E.D.
Similar proof holds for integrals 2, 3, and 4. * For an example of Jordan's inequality, see p.1220, Wylie

\[ \int_0^\infty \cos x^2 \, dx = \int_0^\infty \sin x^2 \, dx = \frac{\sqrt{2\pi}}{4} \] (Fresnel's integral)

**Thm 3** If on the circular arc \( C_\rho \) with radius \( \rho \) and center at \( z = a \),

\( (z-a)f(z) \to 0 \) as \( \rho \to 0 \), then

\[
\lim_{\rho \to 0} \int_{C_\rho} f(z) \, dz = 0
\]

pf: \( |(z-a)f(z)| \leq K_\rho \to 0 \) as \( \rho \to 0 \) \implies \( |f(z)| \leq K_\rho / \rho = M \)

\[
I_\rho \leq \int_{C_\rho} |f(z)| \, dz \leq \frac{K_\rho}{\rho} \cdot \alpha \rho = \alpha K_\rho \to 0 \quad \text{as} \quad \rho \to 0
\]

\( \text{Q.E.D} \)

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Thm 4 Suppose \( f(z) \) has a simple pole at \( z = a \) with \( \text{Res } f(a) \). Then, if \( C_\rho \) is a circular arc with radius \( \rho \) and center at \( z = a \), intercepting an angle \( \alpha \) at \( z = a \), then

\[
\lim_{\rho \to 0} \int_{C_\rho} f(z)\,dz = \alpha i \text{Res } f(a)
\]

pf:

\[
f(z) = \frac{\text{Res } f(a)}{z-a} + \phi(z) \Rightarrow \int_{C_\rho} f(z)\,dz = \int_{C_\rho} \frac{\text{Res } f(a)}{z-a}\,dz + \int_{C_\rho} \phi(z)\,dz
\]

(1)

Let \( z - a = \rho e^{i\theta} \)

\[
(1) = \text{Res } f(a) \int_{\theta_0}^{\theta_0 + \alpha} \frac{\rho e^{i\theta}}{\rho e^{i\theta}} d\theta = i \text{Res } f(a) \int_{\theta_0}^{\theta_0 + \alpha} d\theta = \alpha i \text{Res } f(a)
\]

(2) \( \to 0 \) as \( \rho \to 0 \) (by Thm 3, \( \phi(z) \) is analytic \( \Rightarrow \phi(z) < \infty \) in the neighborhood of \( z = a \) \( \Rightarrow (z-a)\phi(z) \to 0 \) as \( \rho \to 0 \))

Thus, \( \lim_{\rho \to 0} \int_{C_\rho} f(z)\,dz = \alpha i \text{Res } f(a) \quad \text{Q.E.D.} \)

* If \( C_\rho \) is a full circle, \( \lim_{\rho \to 0} \int_{C_\rho} f(z)\,dz = 2\pi i \text{Res } f(a) \) (residue thm)
Indented Contours

→ **Poles** located at the **contour** of integration

For example, a simple pole at $x = x_0$, by Thm4

\[ -\frac{1}{2} \oint_{C_\delta} f(z) \, dz \quad \text{and} \quad \frac{1}{2} \oint_{C_\delta} f(z) \, dz \]

\[ = -\pi i \text{Res} f(x_0) \quad \text{and} \quad = \pi i \text{Res} f(x_0) \]

→ **Indented** around the singularity $x = x_0$ as $\delta \to 0$
\[
\oint_C f(z)\,dz = \int_{-\infty}^{x_0-\delta} f(x)\,dx + \int_{C_{x_0}} f(z)\,dz + \int_{x_0+\delta}^{\infty} f(x)\,dx + \int_{C_{\infty}} f(z)\,dz
\]
\[
= 2\pi i \sum \text{Res (interior)}
\]
\[
= \int_{-\infty}^{x_0-\delta} f(x)\,dx - \pi i \text{Res } f(x_0) + \int_{x_0+\delta}^{\infty} f(x)\,dx + \int_{C_{\infty}} f(z)\,dz
\]
\[
\oint_C f(z)dz = \int_{-\infty}^{x_0-\delta} f(x)dx + \int_{C_{x_0}} f(z)dz + \int_{x_0+\delta}^{\infty} f(x)dx + \int_{C_\infty} f(z)dz
\]

\[
= 2\pi i \left[ \text{Res} f(x_0) + \sum \text{Res(interior)} \right]
\]

\[
= \int_{-\infty}^{x_0-\delta} f(x)dx + \pi i \text{Res} f(x_0) + \int_{x_0+\delta}^{\infty} f(x)dx + \int_{C_\infty} f(z)dz
\]

\[
\Rightarrow 2\pi i \sum \text{Res(interior)}
\]

\[
= \int_{-\infty}^{x_0-\delta} f(x)dx - \pi i \text{Res} f(x_0) + \int_{x_0+\delta}^{\infty} f(x)dx + \int_{C_\infty} f(z)dz
\]

\[
\Rightarrow \text{identical to the previous result (indented upward)}
\]
Cauchy principal value

\[ 2\pi i \sum \text{Res(interior)} = \]

\[ \int_{-\infty}^{x_0-\delta} f(x) \, dx - \pi i \text{Res } f(x_0) + \int_{x_0+\delta}^{\infty} f(x) \, dx + \int_{C_{\infty}} f(z) \, dz \]

\[ \Rightarrow P \int_{-\infty}^{\infty} f(x) \, dx \]

\[ \triangleq \lim_{\delta \to 0} \left\{ \int_{-\infty}^{x_0-\delta} f(x) \, dx + \int_{x_0+\delta}^{\infty} f(x) \, dx \right\} \]

\[ = 2\pi i \sum \text{Res } f(\text{interior}) \]

\[ + \pi i \text{Res } f(x_0) - \int_{C_{\infty}} f(z) \, dz \]

The symmetric interval relative to \( x_0 \) provides cancellation of the shaded area (Cauchy principal value). The contribution of the singularity lies in the integration about the semicircle.
Ex  \[ I = \int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx \] = ?

Sol:

\[ f(z) = \frac{e^{iz}}{z} \]

\[ \oint_{C} f(z) \, dz = \int_{-\rho}^{R} \frac{e^{ix}}{x} \, dx + \int_{C_{\rho}} \frac{e^{iz}}{z} \, dz + \int_{C_{R}} \frac{e^{iz}}{z} \, dz = 0 \quad (\because \text{no pole inside}) \]

Let \[ \tilde{f}(z) = \frac{1}{z} \]

Recall Thm 2, Jordan's lemma, if \( \tilde{f}(z) \to 0 \) as \( R \to \infty \), then

\[ \lim_{R \to \infty} \int_{C_{R}} e^{imz} \tilde{f}(z) \, dz = 0 \quad (m > 0) \]
In our case, $m = 1$, \[ |\tilde{f}(z)| = \left| \frac{1}{z} \right| = \frac{1}{R} \to 0 \quad \text{as} \quad R \to \infty \]

\[
\lim_{R \to \infty} \left\{ \int_{-R}^{-\rho} \frac{e^{ix}}{x} \, dx - \pi i \, \text{Res} \left( \frac{e^{iz}}{z} \right) + \int_{\rho}^{R} \frac{e^{ix}}{x} \, dx \right\} + \oint_{C_{R}} f(z) \, dz = 0
\]

Thus, $P\int_{-\infty}^{\infty} \frac{e^{ix}}{x} \, dx = \pi i \left[ \frac{e^{iz}}{(z)'} \right]_{z=0} = \pi i$ \hspace{1cm} \( \therefore \) Jordan's lemma

$P\int_{-\infty}^{\infty} \left( \frac{\cos x}{x} + i \frac{\sin x}{x} \right) \, dx = \pi i$

$P\int_{-\infty}^{\infty} \frac{\cos x \, dx}{x} = 0$ \hspace{1cm} \( \because \) odd function

$P\int_{-\infty}^{\infty} \frac{\sin x \, dx}{x} = \pi$

$\Rightarrow \int_{0}^{\infty} \frac{\sin x \, dx}{x} = \frac{\pi}{2}$ \hspace{1cm} \( \because \) even function
Integrals Involving Branch Points:

(1) Use an appropriate branch consistently.
(2) The integral along one side of the cut line will probably not equal the integral along the other side because of the discontinuity of the branch cut.
Ex \( I = \int_{0}^{\infty} \frac{x^{m-1}}{x+1} \, dx \) \((0 < m < 1)\)

Sol:

Let \( f(z) = \frac{z^{m-1}}{z+1} \)

\( z^c = e^{\ln z^c}, c \) noninteger, is multi-valued.

\( \{\therefore \ln z = \ln r + i(\theta + 2n\pi)\}, p.756,\text{Kreyszig}\} \)

Choose the branch \( z = re^{i\theta}, 0 \leq \theta < 2\pi \)

\( z^{m-1} = r^{m-1}e^{i(m-1)\theta}, \theta = 0, 2\pi^- \) at the cut

\[
\int_{0}^{\infty} \frac{r^{m-1}}{r+1} \, dr + \oint_{C_R} f(z) \, dz + \int_{-\infty}^{0} \frac{r^{2\pi i}}{re^{2\pi i} + 1} \, d(rie^{2\pi i}) + \oint_{C_R} f(z) \, dz = 2\pi i \text{Res} f(-1)
\]

(\( \theta = 0 \)) \((\therefore \text{Thm 1})\)

(\( \theta = 2\pi^- \)) \((\therefore \text{Thm 3})\)

\(* \oint_{C_R} f(z) \, dz \to 0 \therefore \frac{z^{m-1}}{z+1} = \frac{z^m}{z+1} \to 0 \text{ as } R \to \infty \) (Thm 1)

\( \oint_{C_r} f(z) \, dz \to 0 \therefore (z-0) \frac{z^{m-1}}{z+1} = \frac{z^m}{z+1} \to 0 \text{ as } r \to 0 \) (Thm 3)
\[
\text{Res } f(-1) = \left. \frac{z^{m-1}}{(z+1)} \right|_{z=-1} = (-1)^{m-1} = e^{\pi i}^{m-1}
\]

\[
\int_0^\infty \frac{r^{m-1}}{r+1} dr + \int_0^\infty \frac{(re^{2\pi i})^{m-1}}{re^{2\pi i}+1} d(re^{2\pi i}) = 2\pi i(e^{\pi i})^{m-1}
\]

\[
\int_0^\infty \frac{r^{m-1}}{r+1} dr + \int_0^\infty \frac{r^{m-1} e^{2(m-1)\pi i}}{r+1} e^{2\pi i} dr = 2\pi i(e^{\pi i})^{m-1}
\]

\[
\int_0^\infty \frac{r^{m-1}}{r+1} dr - \int_0^\infty \frac{r^{m-1} e^{2m\pi i}}{r+1} dr = 2\pi i(e^{\pi i})^{m-1}
\]

\[
\Rightarrow (1 - e^{2m\pi i}) \int_0^\infty \frac{r^{m-1}}{r+1} dr = 2\pi i e^{(m-1)\pi i}
\]

\[
\Rightarrow \int_0^\infty \frac{x^{m-1}}{x+1} dx = \frac{2\pi i e^{(m-1)\pi i}}{1-e^{2m\pi i}} = \frac{2\pi i e^{\pi i}}{e^{2m\pi i} - e^{-m\pi i}} = \frac{2\pi i}{e^{m\pi i} - e^{-m\pi i}} = \frac{\pi}{e^{m\pi i} - e^{-m\pi i}} = \frac{\pi}{\sin m\pi}
\]

\[
* I = \int_0^\infty x^{m-1} Q(x) dx = \frac{\pi}{\sin m\pi} \sum \text{Res}\left\{(-z)^{m-1} Q(z)\right\}
\]
HW Wylie

- ch 17.3: 14, 25
- ch 17.4: 1(a), 2(a)
- ch 17.5: 8, 25
- ch 17.6: 1, 4, 15
- ch 18.4: 2, 5
- ch 18.5: 1, 4, 11
- ch 19.1: 3, 9(b), (d), 23
- ch 19.2: 1, 7, 18
Inverse Laplace Transform

\[ \phi(s) = L\{f(t)\} = \int_0^\infty f(t) e^{-st} \, dt \]

\[ f(t) = L^{-1}\{\phi(s)\} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \phi(s) e^{st} \, ds \quad \text{(Bromwich integral)} \]

Thm 1

If the Laplace transform \( \phi(s) \) is an analytic function of \( s \) except at a finite number of poles each of which lies to the left of the vertical line \( \text{Re}(s) = a \), and if \( s\phi(s) < \infty \) as \( s \to \infty \) throughout \( \text{Re}(s) \leq a \), then

\[ L^{-1}\{\phi(s)\} = \sum \text{Res}\{\phi(s)e^{st}\} \]

at each of \( \phi' \)s poles.
Pf:

$$|s\phi(s)| < \infty \quad \text{as} \quad s \to \infty$$

$$\Rightarrow \phi(s) \to 0 \quad \text{uniformly as} \quad R \to \infty$$

$$|\phi(s)| \leq K_R \to 0 \quad \text{as} \quad R \to \infty$$

$$\oint_{ABCDEA} e^{st} \phi(s) \, ds = \int_{AB} e^{st} \phi(s) \, ds + \int_{BC+EA} e^{st} \phi(s) \, ds + \int_{CDE} e^{st} \phi(s) \, ds$$

$$= 2\pi i \sum \text{Res}\{e^{st} \phi(s)\}$$

But,

$$\left| \int_{BC} e^{ts} \phi(s) \, ds \right| \leq \int_{BC} |e^{ts}| |\phi(s)| \, ds = \int_{BC} |e^{t(x+iR)}| |\phi(s)| \, ds$$

$$\left( 0 \leq x \leq a \Rightarrow e^{tx} \leq e^{ta} \right)$$

$$\leq \int_{BC} e^{ta} K_R |ds| = e^{ta} K_R \overline{BC} \to 0 \quad \text{as} \quad R \to \infty$$

A similar argument applies to $EA$.  

---

Mingsian R. Bai
In addition,
\[
\int_{CDE} e^{ts} \phi(s) \, ds \to 0 \text{ as } R \to \infty \text{ by virtue of Jordan's lemma (3)}. 
\]

Specifically, if \( \phi(x) \to 0 \) as \( R \to \infty \),

then
\[
\lim_{R \to \infty} \int_C e^{ts} \phi(s) \, ds = 0 \quad (t > 0) 
\]

Finally, as \( R \to \infty \)

\[
\oint_{ABCDEA} e^{st} \phi(s) \, ds = \int_{AB} e^{st} \phi(s) \, ds + \int_{BC+EA} e^{st} \phi(s) \, ds + \int_{CDE} e^{st} \phi(s) \, ds = 2\pi i \text{Res} \{e^{st} \phi(s)\} 
\]

\[
\Rightarrow L^{-1}\{\phi(s)\} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \phi(s) e^{st} \, ds = \text{Res} \{e^{st} \phi(s)\} 
\]

Q.E.D.
Ex  \( \phi(s) = \frac{1}{(s + a)^2 + b^2} \),  \( L^{-1}\{\phi(s)\} = \) ?

Sol:

Two simple poles: \(-a + ib, -a - ib\)

\[
\begin{align*}
\text{Res } \{e^{st} \phi(s)\} &\left|_{s=-a+ib} \right. = \frac{e^{st}}{s - (-a - ib)} = \frac{e^{(-a+ib)t}}{2ib} \\
\text{Res } \{e^{st} \phi(s)\} &\left|_{s=-a-ib} \right. = \frac{e^{st}}{s - (-a + ib)} = \frac{e^{(-a-ib)t}}{-2ib}
\end{align*}
\]

Thus, \( L^{-1}\{\phi(s)\} = \sum \text{Res}\{\phi(s)e^{st}\} = \frac{e^{(-a+ib)t}}{2ib} + \frac{e^{(-a-ib)t}}{-2ib} \)

\[
= e^{-at} \frac{e^{ibt} - e^{-ibt}}{2ib} = e^{-at} \frac{\sin bt}{b}
\]

→ identical to the result obtained using table lookup!
Stability Criteria

Bounded Input Bounded Output (BIBO) stability

Thm 1

In order for the function \( y(t) = L^{-1}\left\{ \frac{P(s)}{Q(s)} \right\} \) to be stable, it is necessary and sufficient that the equation \( Q(s) = 0 \) has no roots to the right of the imaginary axis in the complex \( s \)-plane, i.e. poles should be located in the LHP, and that any roots on the imaginary axis in the \( s \)-plane be unrepresented.

Pf: \[ L^{-1}\left\{ \frac{1}{s-a} \right\} = e^{at} = e^{(\sigma + i\omega)t} = e^{\sigma t} e^{i\omega t}, \sigma \leq 0 \text{ for stability} \]
Thm 2

If the real part of each root of the polynomial equation \( Q(s) = 0 \) is less than or equal to zero \((\leq 0)\), then the coefficients in \( Q(s) \) all have the same sign.

Pf: Factors \( s + a_i \) and \( (s + a_j)^2 + b_j^2 \), where \( a_i, a_j \geq 0 \) establishes the assertion. (This is only a necessary condition, but it becomes sufficient for quadratic equations.)

Determination of system stability:
1. Routh-Hurwitz stability criterion. (for polynomials only)
2. Nyquist stability criterion. (applicable to more general cases)
Thm If \( f(z) \) is analytic within and on a closed curve \( C \) except at a finite number of poles, and if \( f(z) \) has neither poles nor zeros on \( C \), then
\[
\frac{1}{2\pi i} \oint_C f'(z) \frac{dz}{f(z)} = N - P
\]
where \( N \) is the number of zeros of \( f(z) \) within \( C \) and \( P \) is the number of poles of \( f(z) \) within \( C \), each counted as many times as its multiplicity.

Pf: If \( f(z) \) has a zero of order \( n_k \) at \( z = a_k \)
\[
f(z) = (z - a_k)^{n_k} \phi(z)
\]
\[
f'(z) = n_k (z - a_k)^{n_k-1} \phi(z) + (z - a_k)^{n_k} \phi'(z)
\]
\[
\Rightarrow \frac{f'(z)}{f(z)} = \frac{n_k}{z - a_k} + \frac{\phi'(z)}{\phi(z)} \Rightarrow \text{Res}_{z=a_k} \left\{ \frac{f'(z)}{f(z)} \right\} = n_k
\]
Hence, \( f'(z)/f(z) \) has a simple pole with residue \( (n_k) \) at \( z = a_k \).
On the other hands, if $f(z)$ has a pole of order $p_k$ at $z = b_k$.

$$f(z) = \frac{c_{-p_k}}{(z-b_k)^{p_k}} + \frac{c_{-p_k+1}}{(z-b_k)^{p_k-1}} + \cdots + \frac{c_{-1}}{z-b_k} + c_0 + \cdots$$

$$= (z-b_k)^{-p_k} \psi(z)$$

$$f'(z) = -p_k (z-b_k)^{-p_k-1} \psi(z) + (z-b_k)^{-p_k} \psi'(z)$$

$$\Rightarrow \frac{f'(z)}{f(z)} = \frac{-p_k}{z-b_k} + \frac{\psi'(z)}{\psi(z)} \Rightarrow \text{Res}_{z=b_k} \left\{ \frac{f'(z)}{f(z)} \right\} = -p_k$$

Hence, $f'(z)/f(z)$ has a simple pole with residue ($-p_k$) at $z = b_k$.

By residue theorem,

$$\oint_C \frac{f'(z)}{f(z)} \, dz = 2\pi i \sum \text{residue} = 2\pi i \left( \sum n_k - \sum p_k \right) = 2\pi i (N - P)$$

Q.E.D.
Further interpretation:

$$\frac{1}{2\pi i} \oint_{c} \frac{f'(z)}{f(z)} \, dz = \frac{1}{2\pi i} \oint_{c} d \left[ \ln f(z) \right] = \frac{1}{2\pi i} \left[ \ln |f(z)| + i \angle f(z) \right]_{c}$$

= net variation of $\angle f(z)$ around $C/2\pi$

Thus, we have the following principle of the argument

$$N - P = \frac{\text{variation of } \angle f(z) \text{ around } C}{2\pi}$$

= number of encirclement of $f(z)$ about the origin

If $f(z)$ is analytic everywhere within and on the close curve $C$ (so that $P = 0$) and does not vanish on $C$, then number of zeros of $f(z)$ within $C$

$$N = \frac{\text{variation of } \angle f(z) \text{ around } C}{2\pi}$$

An easy way to count # encirclement:

1. Draw a straight line from the origin
2. # encirclement = # crossings - # reverse crossings
If \( w = Q(z) \) has no roots in the RHP, or \( N = 0 \), then the image curve \( Q(z) \) does not encircle the origin.

The labor of plotting \( w = Q(z) \) can be reduced by letting \( R \to \infty \). The image of the semicircle recedes to \( \infty \) in the \( w \)-plane.

On the semicircle, we have \( z = Re^{i\theta}, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \).
Note that, for the image curve $Q(z)$ of degree $n$,

$$Q(z) = Q(Re^{i\theta}) \sim R^n a_0 e^{in\theta}$$

$\theta$ varies from $-\pi/2$ to $\pi/2 \Rightarrow Q(Re^{i\theta})$ varies from $-n\pi/2$ to $n\pi/2$

$\Rightarrow$ Net variation of $\angle Q(z) = n\pi$

Thus, if $w = Q(z)$ is plotted for $z = i\infty \rightarrow -i\infty$ along the imaginary axis with its net change $\angle w$, then the number of encirclement around the origin of the $w$-plane, or the number of roots in right-half of the $z$ plane is $\frac{\angle w + n\pi}{2\pi}$.

In addition, for a real-coefficient $Q(z)$, $Q(\overline{z}) = \overline{Q(z)}$, $Q(-i\omega) = \overline{Q(i\omega)}$, the image curve of $Q(z)$ for the $z$ values on the lower half of the imaginary axis is just the reflection in the real axis of that for the upper half of the imaginary axis (symmetrical about the real axis).
Ex.

\[ G(s) = \frac{s^2 + 1}{s^3 + s^2 + 4s + 1} \]

**Sol:**

Let \( w = Q(s) = s^3 + s^2 + 4s + 1 \)

On the imaginary axis, \( s = i\omega, \omega = \infty \rightarrow -\infty \), plot \( w = Q(i\omega) \).

\( Q(i\omega) \) does not pass the origin, so \( G(s) \) has no poles on the \( i\omega \) axis. Net variation of \( \arg w \) on the imaginary axis \( i\omega, \omega = \infty \rightarrow -\infty \), is

\[ \angle w = -3\pi/2 - (3\pi/2) = -3\pi \]

By our earlier discussion, this is to be added to argument variation on the semicircle of the contour \((n\pi)\). Thus, the net encirclement is

\[ N = \frac{\angle w + n\pi}{2\pi} = \frac{-3\pi + 3\pi}{2\pi} = 0 \Rightarrow \text{No RHP roots, stable!} \]
Application: feedback control system

The close-loop transfer function $G_o(s) = \frac{L\{x_o\}}{L\{x_i\}} = \frac{G_1(s)}{1+G_1(s)G_2(s)}$

Plot $w = 1 + G_1(s)G_2(s)$ and observe its encirclement around $w = 0$.

Or, equivalently, plot the open-loop transfer function $w = G_1(s)G_2(s)$ and observe its encirclement around $w = -1$.

The close-loop stability can be predicted by plotting the open-loop frequency response function.
Thm **Nyquist stability criterion**

Assume the closed-loop transfer function $G_o(s)$ has no zeros on the imaginary axis. The closed-loop system is stable iff, in the Nyquist plot (FRF of $G_1(i\omega)G_2(i\omega)$) of the open-loop transfer function $G_1(s)G_2(s)$, the number of CW encirclement of $w = -1$ equals the number of open RHP (unstable) poles of $G_1(s)G_2(s)$.

**Pf:**

Let $N, Z, P$ be the number of encirclement of $w = 1 + G_1(s)G_2(s)$ around the origin, the numbers of the open RHP zeros and poles of $w = 1 + G_1(s)G_2(s)$.

Let $G_1(s)G_2(s) = \frac{N(s)}{D(s)}$

$$1 + G_1(s)G_2(s) = 1 + \frac{N(s)}{D(s)} = \frac{D(s) + N(s)}{D(s)} \quad \text{iff} \quad 1 + G_1(s)G_2(s) = 0 \iff G_1(s)G_2(s) = -1$$
That is, \( G_1(s)G_2(s) \) and \( 1 + G_1(s)G_2(s) \) have the same denominator and thus the same number of open RHP poles \( P \). The principle of argument states that \( N = Z - P \). Hence, \( G_o(s) \) is stable, or \( w = 1 + G_1(s)G_2(s) \) has no open RHP zeros iff \( Z = 0 \) or \( N = -P \). Because the \( z \)-curve is chosen in the CCW direction, the stability condition requires the encirclement to be in the CW direction.

Q.E.D.

**Note** Stable open-loop systems

In many cases, the open-loop transfer functions \( G_1(s)G_2(s) \) are stable, i.e., \( P = 0 \). Hence, close-loop stability requires that the Nyquist plot of \( G_1(s)G_2(s) \) does not encircle the point \(-1\). Only experimental data of the frequency response \( G_1(i\omega)G_2(i\omega) \) are needed in these cases, as \( s \) traverses the imaginary axis.
* Homework due next Wednesday (6/17)

* Final exam:
  • Time: 6/24 (GH, Wed)
  • Scope:
    -- Complex variables (4 problems)
    -- Special functions (1 problem)

X Omit everything from this point on.
IV. Conformal mapping

\[ w = f(z) \]

\[ u(x, y) + iv(x, y) = f(x + iy) \]

\[ x + iy \rightarrow u + iv \]
One-to-one mapping with a single-valued inverse of an analytic function \( f(z) \) requires that the transformation has a nonzero Jacobian:

**Jacobian**

\[
J \left( \frac{u, v}{x, y} \right) = \begin{vmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{vmatrix} = \begin{vmatrix}
\frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} \\
\frac{\partial v}{\partial x} & \frac{\partial u}{\partial x}
\end{vmatrix} \quad \text{(Cauchy-Riemann condition)}
\]

\[
= \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2 = \left| \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right|^2 = \left| f'(z) \right|^2 \neq 0
\]
Thm 1  If $f(z)$ is analytic, $w = f(z)$ will have a single-valued inverse at wherever $|f'(z)|^2 \neq 0$

* A point where $f'(z) = 0$ is called a "critical point".
\[ f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \left| \frac{\Delta w}{\Delta z} \right| e^{i\phi} = \lim_{\Delta z \to 0} \left| \frac{\Delta w}{\Delta z} \right| e^{i(\phi - \theta)} = |f'(z)| e^{i(\phi - \theta)}, \]

where \( \phi - \theta = \Delta f'(z) \)

\[ \Delta w = \Delta z |f'(z)| e^{i(\phi - \theta)} \]

\[ |\Delta w| = |f'(z)| |\Delta z|, \quad |f'(z)|: \text{magnification ratio} \]

\[ \Delta \Delta w = \Delta \Delta z + \Delta f'(z) \]

\[ \Rightarrow \Delta \Delta w_1 - \Delta \Delta w_2 = \Delta \Delta z_1 - \Delta \Delta z_2 \]
Thm

In the mapping defined by an analytic function \( w = f(z) \), \( |f'(z)|^2 \neq 0 \), then

(1) **length**: \( |\Delta w| = |f'(z)||\Delta z| \),

(2) **area**: \( area_1 = |f'(z)|^2 \cdot area_2 \)

(3) **angle**: \( \Delta w_1 - \Delta w_2 = \Delta z_1 - \Delta z_2 \)

\( \Rightarrow \) (1) These conclusions hold for only infinitesimal scales.
(2) Angle preservation (except at critical points)
→ Conformal mapping
Suppose \( f'(z) = 0 \) has a \( n \)-fold zero at \( z = z_0 \)

\[
f'(z) = (n+1)a(z - z_0)^n + (n+2)b(z - z_0)^{n+1} + \cdots
\]

\[
\Rightarrow f(z) = f(z_0) + a(z - z_0)^{n+1} + b(z - z_0)^{n+2} + \cdots
\]

\[
z - z_0 = \Delta z, \quad f(z) - f(z_0) = \Delta w = a(\Delta z)^{n+1} + b(\Delta z)^{n+2} + \cdots
\]

\[
\Rightarrow \frac{\Delta w}{a(\Delta z)^{n+1}} = \frac{a(\Delta z)^{n+1} + b(\Delta z)^{n+2} + \cdots}{a(\Delta z)^{n+1}} = 1 + \frac{b}{a} \Delta z + \cdots
\]

As \( \Delta z \to 0 \),

\[
\frac{\Delta w}{a(\Delta z)^{n+1}} \to 1 = e^{i0} \Rightarrow \lim_{\Delta z \to 0} \Delta w - \lim_{\Delta z \to 0} a(\Delta z)^{n+1} = 0
\]

\[
\Delta w = \Delta a + (n+1) \Delta z \Rightarrow \Delta w_1 - \Delta w_2 = (n+1) \left( \Delta z_1 - \Delta z_2 \right)
\]
**Theorem** Harmonic functions remain harmonic under conformal mapping,

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{where} \quad w = f(z)
\]

where \( w = f(z) \) or \( u + iv = f(x + iy) \) is an analytic function.

**Proof:**

\[\phi(x, y) \xrightarrow{w = f(z)} \phi(u, v)\]

\[\phi_x = \phi_u u_x + \phi_v v_x \quad \text{and} \quad \phi_y = \phi_u u_y + \phi_v v_y\]

\[\phi_{xx} = \frac{\partial \phi_u}{\partial x} u_x + \phi_u u_{xx} + \frac{\partial \phi_v}{\partial x} u_x + \phi_v u_{xx}\]

\[= \left( \phi_{uu} u_x + \phi_{uv} u_x \right) u_x + \phi_u u_{xx} + \left( \phi_{vu} u_x + \phi_{vv} v_x \right) v_x + \phi_v v_{xx}\]

\[\phi_{yy} = \left( \phi_{uu} u_y + \phi_{uv} v_y \right) u_y + \phi_u u_{yy} + \left( \phi_{vu} u_y + \phi_{vv} v_y \right) v_y + \phi_v v_{yy}\]
\[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \phi_{xx} + \phi_{yy} \]

\[ = \phi_u \left( u_{xx} + u_{yy} \right) + \phi_{uu} \left( u_x^2 + u_y^2 \right) + 2\phi_{uv} \left( u_x v_x + u_y v_y \right) + \phi_v \left( v_{xx} + v_{yy} \right) + \phi_{vv} \left( v_x^2 + v_y^2 \right) \]

\[ \because \text{Laplace eq.} \quad \therefore \text{Cauchy-Riemann} \quad \therefore \text{Laplace eq.} \]

\[ \Rightarrow \phi_{xx} + \phi_{yy} = \phi_{uu} \left[ u_x^2 + (-v_x)^2 \right] + \phi_{vv} \left[ v_x^2 + u_x^2 \right] \]

\[ = (u_x^2 + v_x^2)(\phi_{uu} + \phi_{vv}) = \left| \frac{\partial}{\partial x} (u + iv) \right|^2 (\phi_{uu} + \phi_{vv}) \]

\[ \because f(z) \text{ is a analytic function} \]

\[ = \left| f'(z) \right|^2 \left( \frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2} \right) \]

\[ \because f(z) \text{ is a conformal mapping, } \left| f'(z) \right|^2 \neq 0 \]

\[ \therefore \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \Rightarrow \frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2} = 0 \]

Q.E.D.
If a direct attack on the original problem is difficult, then it may be easier to find a conformal mapping which converts $R$ into some simpler $R'$ in which $\nabla^2 \phi = 0$ can be solved st. the transformed BC.
Thm1 Bilinear Transformation

\[ w = \frac{az + b}{cz + d}, \quad ad - bc \neq 0 \] transforms a circle/line to a circles/line

Pf: Consider the following transformations:

a. \( w = z + \lambda \)

b. \( w = \mu z \)

c. \( w = \frac{1}{z} \)

Case a. \( w = z + \lambda \) is just a translation.

Case b. \( w = \mu z \) : circle to circle

Circle: \( a(x^2 + y^2) + bx + cy + d = 0, \ b^2 + c^2 \geq 4ad \)

\[ z = x + iy \Rightarrow x = \frac{z + \bar{z}}{2}, \ y = \frac{z - \bar{z}}{2i}, \ x^2 + y^2 = |z|^2 = z \bar{z} \]

\[ \Rightarrow az\bar{z} + \frac{b - ic}{2} z + \frac{b + ic}{2} \bar{z} + d = 0 \]
Rewrite this into
\[
(A + \bar{A}) z\bar{z} + Bz + \bar{B}\bar{z} + (D + \bar{D}) = 0, \quad B\bar{B} \geq (A + \bar{A})(D + \bar{D})
\]
Now, \( z = w/\mu \)
\[
(A + \bar{A}) \frac{w}{\mu} \frac{\bar{w}}{\bar{\mu}} + B \frac{w}{\mu} + \bar{B} \frac{\bar{w}}{\bar{\mu}} + (D + \bar{D}) = 0
\]
or \[
(A + \bar{A}) w\bar{w} + (B\bar{\mu}) w + (\bar{B}\mu) \bar{w} + (D + \bar{D}) \mu\bar{\mu} = 0 \rightarrow \text{circle again!}
\]
Note, if \( A + \bar{A} = 0 \rightarrow bx + cy + d = 0 \), it is a straight line!

Case c. \( w = \frac{1}{z} \): circle to circle
\[
(A + \bar{A}) z\bar{z} + Bz + \bar{B}\bar{z} + (D + \bar{D}) = 0
\]
\[
(A + \bar{A}) \frac{1}{w} \frac{1}{\bar{w}} + B \frac{1}{w} + \bar{B} \frac{1}{\bar{w}} + (D + \bar{D}) = 0
\]
\[
(D + \bar{D}) w\bar{w} + \bar{B}w + B\bar{w} + (A + \bar{A}) = 0 \rightarrow \text{circle again!}
\]
Chain of transformations:

\[ w_1 = z + \frac{d}{c} \quad (a) \]
\[ w_2 = cw_1 = cz + d \quad (b) \]
\[ w_3 = \frac{1}{w_2} = \frac{1}{cz + d} \quad (c) \]
\[ w_4 = \frac{bc - ad}{c} w_3 = \frac{bc - ad}{c(cz + d)} \quad (b) \]
\[ w = w_4 + \frac{a}{c} = \frac{bc - ad}{c(cz + d)} + \frac{a}{c} = \frac{az + b}{cz + d} \quad (a) \]

These composite operations transform a circle/line to a circle/line. On the other hand, if \( c = 0 \)
\[ w_1 = z + \frac{b}{a} \quad (a) \]
\[ w = \frac{a}{d} w_1 = \frac{a}{d} \left( z + \frac{b}{a} \right) = \frac{az + b}{d} \quad (b) \]
The conclusion remains the same.
Thm2 The following cross-ratio is invariant under a bilinear transformation.

\[
\frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}
\]

pf: see p.996, Wylie.

* Three conditions, e.g., \(\frac{b}{a}, \frac{c}{a}, \frac{d}{a}\) determine a bilinear transformation.
Ex: Find the bilinear transformation that maps the upper-half plane onto the interior of the unit circle.

Sol:

Boundaries: \( \text{Im}(z) = 0 \rightarrow |w| = 1 \)

\[
|w| = \left| \frac{az + b}{cz + d} \right| = \left| \frac{a}{c} \right| \left| \frac{z + b/a}{z + d/c} \right| = 1
\]

on the unit circle.
As $|z| \to \infty$, $\frac{|a|}{c} = 1 \Rightarrow \frac{a}{c} = e^{i\theta}$, $\forall \text{Im}(z) = 0$, $\left| z + \frac{b}{a} \right| = \left| z + \frac{d}{c} \right|$

$\Rightarrow \left| z - \lambda \right| = \left| z - \bar{\lambda} \right|$ where $\lambda = -\frac{b}{a}$, $\bar{\lambda} = -\frac{d}{c}$

Hence, $w = \frac{az + b}{cz + d} = \frac{a}{c} \frac{z + b/a}{z + d/c} = e^{i\theta} \frac{z - \lambda}{z - \bar{\lambda}}$

This can be verified using a convenient point $z = \lambda \in$ upper half of the $z$-plane, e.g., $z = \lambda$ is mapped into $w = 0$ inside the unit circle.

As a special case, let $e^{i\theta} = -1$, $\lambda = i$, then

$w = -\frac{z - i}{z + i}$

$\Rightarrow \text{Im}\{w\} = \frac{w - \bar{w}}{2i} = -\frac{1}{2i} \left( \frac{z - i}{z + i} - \frac{\bar{z} + i}{\bar{z} - i} \right) = \frac{z + \bar{z}}{(z + i)(\bar{z} - i)} = \frac{2\text{Re}\{z\}}{|z + i|^2}$

$\text{Im}\{w\}$ will have the same sign as $\text{Re}\{z\}$. 

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1st quadrant of the $z$-plane is mapped into the upper half of the circle.  
2nd quadrant of the $z$-plane is mapped into the lower half of the circle.

On the other hand, the inverse transformation

$$z = f^{-1}(w) = -i \frac{w-1}{w+1}$$

maps the interior of the unit circle onto the upper half of the $z$-plane in such a way that the upper half of the circle maps onto the 1st quadrant of the $z$-plane.
Ex.

\[ T = 0 \oplus \frac{T = 100}{b} \]

Heat conduction

\[ w = f(z) \]

\[ u' \]

\[ c' = \infty \]

\[ T = 0 \]

\[ b' \]

\[ 0 \]

\[ o' \]

\[ a' \]

\[ \zeta \text{-plane} \]

\[ w \text{-plane} \]
Sol:

The bilinear transform that maps \((a, b, c)\) to \((a', b', c')\) is \(w = \frac{1 - z}{z}\).

In the \(w\)-plane, \(\nabla^2 \phi(u, v) = 0\) \(\phi\): Temperature

From symmetry, \(\phi(u, v) = \phi(u)\)

\[
\frac{d^2 \phi}{du^2} = 0 \rightarrow \phi = Au + B
\]

BC. \(\phi(0) = 0, \phi(1) = 100\)

\[\Rightarrow \phi = 100u \quad \ast \exists \psi = 100v \odot \Phi = \phi + i\psi\]

\[w = u + iv = \frac{1 - z}{z} = \frac{1}{z} - 1 = \frac{1}{x + iy} - 1\]

\[\Rightarrow u = \frac{x}{x^2 + y^2} - 1, \quad v = \frac{-y}{x^2 + y^2}\]

\[\phi(x, y) = 100u = 100 \left( \frac{x}{x^2 + y^2} - 1 \right)\]
Singularity at $z = 0$ is due to the discontinuous BC.

**Isotherms:**

$$T_0 = 100 \left( \frac{x}{x^2 + y^2} - 1 \right)$$

or

$$\left[ x - \frac{1/2}{1 + T_0/100} \right]^2 + y^2 = \left( \frac{1/2}{1 + T_0/100} \right)^2 \rightarrow \text{a circle}$$

**Heat flux:**

$$q(z) = -k \left( \frac{d\Phi}{dz} \right) = Q_x + iQ_y = 100k \left( \frac{1}{z^2} \right) = \frac{100k}{(z)^2}$$
The Schwarz-Christoffel Transformation:

\[ w = z^m = r^m e^{im\theta} \]

\[ w - w_1 = (z - x_1)^{\alpha_1/\pi} \quad \Rightarrow \quad \frac{dw}{dz} = \frac{\alpha_1}{\pi} (z - x_1)^{(\alpha_1/\pi) - 1} \]
For a polygon,

\[ \frac{dw}{dz} = K \left( z - x_1 \right)^{\left(\frac{\alpha_1}{\pi}\right) - 1} \left( z - x_2 \right)^{\left(\frac{\alpha_2}{\pi}\right) - 1} \cdots \left( z - x_n \right)^{\left(\frac{\alpha_n}{\pi}\right) - 1} \]

\[ \Rightarrow \, \Delta dw = \Delta dK + \left( \frac{\alpha_1}{\pi} - 1 \right) \Delta(z - x_1) + \cdots + \left( \frac{\alpha_n}{\pi} - 1 \right) \Delta(z - x_n) + \Delta dz \]
As $z$ passes $x_1$, $\Delta z - x_1$: from $\pi$ to 0

$$\left(\frac{\alpha_1}{\pi} - 1\right)(-\pi) = \pi - \alpha_1$$

Now, $w = K \int \left[ (z - x_1)^{(\alpha_1/\pi) - 1} (z - x_2)^{(\alpha_2/\pi) - 1} \cdots (z - x_n)^{(\alpha_n/\pi) - 1} \right] dz + C$

* (1) There condition can be arbitrarily specified.

(2) $z = \infty$ can be treated as a vertex.

(3) In general, it's more often polygon $\rightarrow$ line.
Ex

\[ \frac{dz}{dw} = K (w-0)^{\alpha/\pi - 1} = Kw^{\alpha/\pi - 1} + C \Rightarrow z = \frac{K \pi}{\alpha} w^{\alpha/\pi} + C \]

\[ z = 0 \rightarrow w = 0: \quad 0 = C \]

\[ z = 1 \rightarrow w = 1: \quad 1 = \frac{K \pi}{\alpha} \]

\[ z = w^{\alpha/\pi}, \ 0 \leq \Delta z \leq \alpha, \ 0 \leq 4w \leq \pi \]
In the $w$-plane, we have the boundary value problem:

\[
\frac{\partial^2 T}{\partial u^2} + \frac{\partial^2 T}{\partial v^2} = 0 \quad (\infty < u < \infty, \; 0 < v < \infty) \tag{1}
\]

BC: $T_0$ at $v = 0$
Def. The Fourier transform pair,
\[ \tilde{T}(k_u, v) = \int_{-\infty}^{\infty} T(u, v)e^{-ik_uu} \, dk_u \]
\[ T(u, v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{T}(k_u, v)e^{ik_uu} \, dk_u \]
Transformation (1) gives
\[ (ik_u)^2 \tilde{T} + \frac{\partial^2 \tilde{T}}{\partial v^2} = 0 \text{ or } \frac{d^2}{dv^2} \tilde{T} - k_u^2 \tilde{T} = 0 \]
BC: \( \tilde{T}_0 (k_u, 0) = \int_{-\infty}^{\infty} T(u, 0)e^{-ik_uu} \, du = \int_{-\infty}^{\infty} T_0 e^{-ik_uu} \, du \)
\[ \Rightarrow \tilde{T}(k_u, v) = Ae^{\mid k_u \mid v} + Be^{-\mid k_u \mid v} \quad (\because -\infty < k_u < \infty) \]
Implicit BC at \( v = \infty \): \( \tilde{T}(k_u, \infty) = 0 \rightarrow A = 0 \)
BC at \( v = 0 \): \( \tilde{T}_0 = B \)
Hence, $\tilde{T}(k_u, v) = T_0 e^{-|k_u|v}$

$\Rightarrow T(u, v) = T_0 \ast F_u^{-1}\left\{ e^{-|k_u|v} \right\}$

But $F_u^{-1}\left\{ e^{-|k_u|v} \right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|k_u|v} e^{ik_uu} dk_u$

$= \frac{1}{2\pi} \int_{-\infty}^{0} e^{k_uv} e^{ik_uu} dk_u + \frac{1}{2\pi} \int_{0}^{\infty} e^{-k_uv} e^{ik_uu} dk_u$

$= \frac{1}{2\pi} e^{(v+iu)k_u} \bigg|_{0}^{0} + \frac{1}{2\pi} e^{(-v+iu)k_u} \bigg|_{-\infty}^{\infty}$

$= \frac{1}{2\pi} \frac{1}{v+iu} - \frac{1}{2\pi} \frac{1}{-v+iu} = \frac{1}{\pi} \frac{v}{u^2 + v^2}$
Thus,

\[ T(u, v) = T_0 * \left( \frac{1}{\pi u^2 + v^2} \right) = \int_{-\infty}^{\infty} \frac{T_0}{\pi} \frac{vd\xi}{(\xi - u)^2 + v^2} = \int_{0}^{1} \frac{1}{\pi} \frac{vd\xi}{(\xi - u)^2 + v^2} \]

\[ = \frac{1}{\pi} \left( \tan^{-1} \frac{1-u}{v} + \tan^{-1} \frac{u}{v} \right) = \frac{1}{\pi} \cot^{-1} \frac{u^2 + v^2 - u}{v} \]

Let \( w = \rho e^{i\phi} \), \( u = \rho \cos \phi \), \( v = \rho \sin \phi \)

\[ T = \frac{1}{\pi} \cot^{-1} \frac{\rho^2 - \rho \cos \phi}{\rho \sin \phi} \]

Inverse \( w \) to the \( z \)-plane and let \( z = re^{i\theta} \), \( x = r \cos \theta \), \( y = r \sin \theta \)

\[ w = z^{\pi/\alpha} \Rightarrow \rho e^{i\phi} = r^{\pi/\alpha} e^{\pi/\alpha i\theta} \Rightarrow \rho = r^{\pi/\alpha}, \ \phi = \frac{\pi \theta}{\alpha} \]

Thus, \( T(r, \theta) = \frac{1}{\pi} \cot^{-1} \left[ \frac{r^{\pi/\alpha} - \cos(\pi \theta/\alpha)}{\sin(\pi \theta/\alpha)} \right] \)