10 Waveguide

10.1 Introduction (also see §9.1)

1. EM wave in free space: unguided, less efficient

2. EM waves can be guided by waveguides or transmission lines for more efficient power transmission

10.2 Uniform Guiding Structures

1. Straight guiding structure with uniform cross section

In phasor form,

The phasor wave equations (Helmholtz’s equation) are,

\[
\begin{align*}
\nabla^2 E + k^2 E &= 0 \\
\nabla^2 H + k^2 H &= 0
\end{align*}
\]

\[ k = \omega \sqrt{\mu \epsilon} = \frac{2\pi}{\lambda} \]

The Laplacian operator \( \nabla \) can be written as

\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} =
\]

\[
\Rightarrow \begin{cases} 
\nabla^2 E + k^2 E = 0 \\
\nabla^2 H + k^2 H = \nabla^2_{xy} H + (\gamma^2 + k^2) H = 0
\end{cases}
\]

where \( k \) = wave number of plane wave along any direction in free space,

\( \gamma \) = wave number of guided wave in waveguides along \( z \).

2. With \( \partial / \partial z \rightarrow -\gamma \), the six differential equations for the field components can be obtained from Maxwell’s equations,
\( \nabla \times \mathbf{E} = -j\omega\mu \mathbf{H} \quad \quad \quad \nabla \times \mathbf{H} = j\omega\varepsilon \mathbf{E} \)

\[ \Rightarrow \begin{cases} \frac{\partial E_y^0}{\partial y} + \gamma E_y^0 = -j\omega\mu H_x^0 \\ -\gamma E_x^0 - \frac{\partial E_x^0}{\partial x} = -j\omega\mu H_y^0 \\ \frac{\partial E_y^0}{\partial x} - \frac{\partial E_x^0}{\partial y} = -j\omega\mu H_z^0 \end{cases} \]

\[ \Rightarrow \begin{cases} \frac{\partial H_x^0}{\partial y} + \gamma H_y^0 = j\omega E_x^0 \\ -\gamma H_x^0 - \frac{\partial H_x^0}{\partial x} = j\omega E_y^0 \\ \frac{\partial H_y^0}{\partial x} - \frac{\partial H_x^0}{\partial y} = j\omega E_y^0 \end{cases} \]

The field components are not all independent. Usually, the transvers field components \((x, y)\) are expressed in terms of the longitudinal components \((z)\):

\[ \Rightarrow \begin{cases} H_x^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial H_x^0}{\partial x} - j\omega\mu \frac{\partial E_x^0}{\partial y} \right) \quad (10-11) \\ H_y^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial H_y^0}{\partial y} + j\omega\mu \frac{\partial E_y^0}{\partial x} \right) \quad (10-12) \\ E_x^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial E_x^0}{\partial y} + j\omega \frac{\partial H_x^0}{\partial y} \right) \quad (10-13) \\ E_y^0 = -\frac{1}{h^2} \left( \gamma \frac{\partial E_y^0}{\partial x} - j\omega \frac{\partial H_y^0}{\partial x} \right) \quad (10-14) \]

where \( h^2 = k^2 + \gamma^2 \).

\( \Rightarrow E_z^0 \) and/or \( H_z^0 \) are first solved from the wave equations and boundary conditions, and \( E_x^0, E_y^0, H_x^0, H_y^0 \) can be obtained from the above equations (Eqs. 10-11 \sim 10-14).

- TEM waves: \( E_z^0 = 0, H_z^0 = 0 \), no longitudinal field components
- TM waves: \( E_z^0 \neq 0, H_z^0 = 0 \), no longitudinal \( \mathbf{H} \) field components
- TE waves: \( E_z^0 = 0, H_z^0 \neq 0 \), no longitudinal \( \mathbf{E} \) field components
10.3 Parallel-Plate Waveguide

- neglect fringing effect ($W \gg b$)
- perfect conductor
- lossless dielectric

1. TEM wave (see §9-2)

Consider a $y$-polarized uniform plane wave propagating in the $+z$ direction between the two conductors

Boundary conditions at $y = 0$ and $y = b$: $E_t = 0$ and $H_n = 0$

⇒ satisfied because of no tangential $E$ and normal $H$ in the above wave

⇒ The plane wave is a solution (mode) in the parallel plate waveguide

- The propagation characteristics ($\gamma$, $\eta$, $u_p$) of a guided TEM mode are similar to those of a plane wave in free space.

- In terms of Eq. 10-2, $E_0(x, y) = constant$.

2. Surface charge and current due to discontinuity of $E_n$ and $H_t$:

On lower plate ($y = 0$), $a_n = a_y$; from boundary conditions,

On upper plate ($y = b$), $a_n = -a_y$,

\[
\begin{cases}
-a_y \cdot D = \rho_{su} \quad \Rightarrow \quad \rho_{su} = -\epsilon E_y = -\epsilon E_0 e^{-j\beta z} \\
-a_y \times H = J_{su} \quad \Rightarrow \quad J_{su} = a_z H_x = -a_z (E_0/\eta) e^{-j\beta z}
\end{cases}
\]
10.3.1 TM waves \((H_z = 0, E_z \neq 0)\)

1. To find a TM wave (mode) in the waveguide,

\[ H_0^z(y) = 0 \text{ at } y = 0 \text{ and } y = b \ (E_z = 0) \]

\[ \Rightarrow \]

\[ \begin{align*}
H_0^y(y) &= -\frac{j\omega \epsilon}{h^2} \frac{\partial E_0^z}{\partial y} = 0 \\
E_0^x(y) &= -\frac{\gamma}{h^2} \frac{\partial E_0^y}{\partial x} = 0 \\
E_0^y(y) &= -\frac{\gamma}{h^2} \frac{\partial E_0^x}{\partial y} = -\frac{\gamma}{h} A_n \cos \frac{n\pi y}{b} \\

\end{align*} \]

\[ \Rightarrow \gamma = \sqrt{h^2 - \omega^2 \mu \epsilon} \]

\[ \Rightarrow \gamma = \]

(a) \( f > f_c, \gamma = j\beta \Rightarrow \text{propagation mode} \)

(b) \( f < f_c, \gamma = \alpha \Rightarrow \text{decay/evanescent mode} \)

Ex 10-3: TM\(_n\) mode \((n = 1)\)

(a) instantaneous expressions

\[ E_x(y, z, t) = \]

\[ E_y(y, z, t) = \cdots = \frac{\beta b}{\pi} A_1 \cos \left( \frac{\pi y}{b} \right) \sin(\omega t - \beta z) \]

\[ H_x(y, z, t) = \cdots = -\frac{\omega \epsilon b}{\pi} A_1 \cos \left( \frac{\pi y}{b} \right) \sin(\omega t - \beta z) \]
(b) field lines

\( \mathbf{E} \) field lines in \( y - z \) plane:

\( \mathbf{H} \) field lines:

\[ \mathbf{H} = \mathbf{H}_x = \text{perpendicular to } yz \text{ plan} \]

\( \text{(} J_s \text{ can be found at } y = 0, b \text{)} \)

Ex 10-4: TM\textsubscript{1} mode as superposition of two plane waves
\[ \Rightarrow k_1, k_2 \text{ bouncing up and down, no net } z \text{ propagation} \]

2. At cutoff \( (\lambda/2b = 1) \)

\[ \rightarrow \text{time harmonic field with no spatial variation in } z \text{ direction} \]

- In the waveguide, \( k_1, k_2 \) form standing wave that satisfies the B.C..
- For \( \lambda > \lambda_c, f < f_c \), standing waves satisfying the B.C. can not be formed.
10.3.2 TE wave \((E_z = 0; H_z \neq 0)\)

1. \(E_z = 0\), from Eq. 10-48,

B.C.: \(E_x = 0\) at conductor surfaces

\[
\begin{align*}
H_0^y(y) &= \frac{\gamma}{n} B_n \sin \frac{n\pi y}{b} \\
E_0^x(y) &= j\omega \mu B_n \sin \frac{n\pi y}{b}
\end{align*}
\]

where \(\gamma = \sqrt{\left(\frac{n\pi}{b}\right)^2 - \omega^2 \mu \epsilon}\).

- \(f_{c,TE} = f_{c,TM} = n/2b\sqrt{\mu\epsilon}\)
- For \(n = 0\), \(H_y = 0, E_x = 0\)
  \(\Rightarrow\) TE\(_0\) mode does not exist in parallel plate waveguides

Ex 10-5: (see textbook) (Fig. 10-8)
10.6 Dielectric waveguide

10.6.1 TM mode in a slab dielectric waveguide

1. For TM modes, $H_z = 0$

EM fields exist in both free space and dielectric slab. Waveguide modes can be found by matching the B.C. for wave solutions of Eq. 10-241 in the three regions.

(a) In the slab, $|y| \leq d/2$,

⇒ solution of Eq. 10-241 is

(b) In free space, $y \geq d/2$ and $y \leq -d/2$, the solution of Eq. 10-241 can be sinusoidal or ± exponential functions. But waveguide modes must be confined to the region near the slab.

⇒
(c) \( k_y, \alpha \): to be determined by B.C.

\( E_o, E_e, C_u, C_l \): to be determined by initial conditions

i. odd TM mode

\( E^0_z(y) \) is described by a sine function and is antisymmetric with respect to the \( y = 0 \) plane.

A. in the dielectric region, \( |y| \leq d/2 \), from Eqs. 10-244, 10-11, 10-14,

\[
\begin{align*}
E^0_z(y) &= \\
E^0_y(y) &= -\frac{j\beta}{k_y} E_0 \cos k_y y \\
H^0_x(y) &= \frac{j\omega \varepsilon_0}{k_y} E_0 \cos k_y y
\end{align*}
\]

B. in the upper free space, \( y \geq d/2 \)

\[
\begin{align*}
E^0_z(y) &= \\
E^0_y(y) &= -\frac{j\beta}{\alpha} E_0 \sin \frac{k_y d}{2} e^{-\alpha (y-d/2/2)} \\
H^0_x(y) &= \frac{j\omega \varepsilon_0}{\alpha} E_0 \sin \frac{k_y d}{2} e^{-\alpha (y-d/2/2)}
\end{align*}
\]

\((C_u = E_0 \sin \frac{k_0 d}{2} \) from B.C.\)

C. in the lower free space, \( y \leq d/2 \)

\[
\begin{align*}
E^0_z(y) &= \\
E^0_y(y) &= -\frac{j\beta}{\alpha} E_0 \sin \frac{k_y d}{2} e^{\alpha (y+d/2/2)} \\
H^0_x(y) &= \frac{j\omega \varepsilon_0}{\alpha} E_0 \sin \frac{k_y d}{2} e^{\alpha (y+d/2/2)}
\end{align*}
\]
D. From B.C., tangential H \( (H_x) \) must be continuous at \( y = \pm d/2 \),
\[ \Rightarrow \text{From Eqs. 10-250, 10-253,} \]
\[ \Rightarrow \text{Eq. 10-260 can be solved numerically or graphically} \]

ii. Similarly, even TM modes can be obtained from

note: - must have \( \mu_d \epsilon_d > \mu_0 \epsilon_0 \) (or \( \epsilon_d > \epsilon_0 \) for non-magnetic materials) to have guided modes
- guided modes are ”discrete”
- when \( \omega \) increases, more modes can be supported
- when \( d \) increases, more modes can be supported
- when \( \mu_d \epsilon_d - \mu_0 \epsilon_0 \) increases, more modes can be supported
- optical fiber is a cylindrical dielectric waveguide