9 Transmission Lines

9.1 Introduction

1. EM wave in free space and in waveguides or transmission lines

2. Examples of transmission lines
   - two-wire lines
   - coaxial cable
   - metal interconnection in PCB or IC

9.3 General Transmission Line Equations

1. Electric circuit: $\ell \ll \lambda$, no current/voltage variation in lumped elements

   Transmission line: $\ell \gg, \geq \lambda$, distributed circuit elements

2. Distributed model of transmission lines

   Consider a differential length $\Delta z$ of a general two conductor transmission line (parallel plate, coaxial cables, etc.) described by
   \[
   \begin{align*}
   R &= \text{resistance per unit length of the conductors (}\Omega/\text{m}) \\
   L &= \text{inductance per unit length of the line (}\text{H/m}) \\
   G &= \text{conductance per unit length of the dielectric medium (}\text{S/m}) \\
   C &= \text{capacitance per unit length of the line (}\text{F/m})
   \end{align*}
   \]

   (see Tables 9-1 and 9-2)
3. Applying Kirchhoff’s voltage law, we have

4. Applying Kirchhoff’s current law at node N, we have

\[ \begin{align*}
\text{(9-31) and (9-33) are the general transmission line equations (telegrapher’s equations)}
\end{align*} \]

5. In phasor form,

\[
\begin{align*}
\begin{cases}
v(z, t) &= \Re[V(z)e^{j\omega t}] \\
i(z, t) &= \Re[I(z)e^{j\omega t}]
\end{cases}
\end{align*}
\Rightarrow
\begin{align*}
-\frac{dV(z)}{dz} &= (R + j\omega L)I(z) \\
-\frac{dI(z)}{dz} &= (G + j\omega C)V(z)
\end{align*}
\tag{9-35}
\]

- time harmonic transmission line equation

- In circuit theory, Fourier transform reduces an ordinary differential equation to an algebraic equation.

In EM and transmission line theory, Fourier transform reduces a partial differential equation to an ordinary differential equation.

### 9.3.1 Wave on an Infinite Transmission Line

1. Decoupled wave equations:

\[
\begin{align*}
\begin{cases}
\gamma &= \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} = \text{propagation constant} \\
\alpha &= \text{attenuation constant} \\
\beta &= \text{phase constant}
\end{cases}
\end{align*}
\]
2. Solution of the wave equations:

For an infinite line, only \( V^+(z) \) and \( I^+(z) \) exist (no reflection)

\[
\begin{cases}
V(z) = V^+(z) = V_0^+ e^{-\gamma z} \\
I(z) = I^+(z) = I_0^+ e^{-\gamma z} \\
Z_0 \triangleq \frac{V(z)}{I(z)} = \frac{V_0^+}{I_0^+} = \text{constant}
\end{cases}
\]

\( \Rightarrow Z_0 = \text{characteristic impedance of the line (independent of length)} \)

Ex 9-2: Analogy between waves on a transmission line and in a dielectric medium

- For a uniform plane wave with \( E_x \) and \( H_y \) only, the Maxwell’s equations can be reduced to

\[
\begin{cases}
\nabla \times E = -j\omega (\mu' - j\mu'') H \\
\nabla \times H = j\omega (\epsilon' - j\epsilon'') E
\end{cases}
\]

\[
\Rightarrow \begin{cases}
E_x \leftrightarrow V \\
H_y \leftrightarrow I
\end{cases} \quad \begin{cases}
R \leftrightarrow \omega \mu'' \\
L \leftrightarrow \mu' \\
G \leftrightarrow \omega \epsilon'' \\
C \leftrightarrow \epsilon'
\end{cases}
\]

\( \gamma = \alpha + j\beta = \omega \sqrt{\mu \epsilon} = \sqrt{\omega \mu'' + j\omega \mu'} (\omega \epsilon'' + j\omega \epsilon') \)

\( Z_0 \leftrightarrow \eta_c = \sqrt{\frac{\mu'' + j\mu'}{\epsilon'' + j\epsilon'}} \)

- Reflection from interface or discontinuity

3. Characteristic impedance and propagation constant

\[
Z_0 = \frac{\sqrt{R + j\omega L}}{G + j\omega C}
\]

\( \gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} \)
(a) lossless lines \((R = 0, G = 0)\)

\[
\begin{align*}
\gamma &= j\omega \sqrt{LC}, \alpha = 0, \beta = \omega \sqrt{LC} \\
u_p &= \omega / \beta = 1/\sqrt{LC} \\
Z_0 &= R_0 + jX_0 = \sqrt{L/C} \Rightarrow \begin{cases} 
R_0 = \sqrt{L/C} = \text{constant} \\
X_0 &= 0
\end{cases}
\end{align*}
\]

(b) low loss lines \((R \ll \omega L, G \ll \omega C)\)

\[
\begin{align*}
\Rightarrow \alpha &\approx \frac{1}{2} \left( R \sqrt{\frac{C}{L}} + G \sqrt{\frac{L}{C}} \right) \\
\beta &\approx \omega \sqrt{LC} \\
u_p &= \omega / \beta \approx 1/\sqrt{LC} \text{ (approximately constant)} \\
Z_0 &= R_0 + jX_0 \approx \sqrt{\frac{L}{C}} \left( 1 + \frac{1}{j2\omega} \left( \frac{R}{L} - \frac{G}{C} \right) \right) \\
\Rightarrow R_0 &\approx \sqrt{\frac{L}{C}}, \\
X_0 &\approx \sqrt{\frac{L}{C}} \frac{1}{2\omega} \left( \frac{R}{L} - \frac{G}{C} \right) \to 0 \text{ for low loss lines}
\end{align*}
\]

(c) distortionless (dispersionless) lines \((\frac{R}{L} = \frac{G}{C})\)

\[
\begin{align*}
\Rightarrow \alpha &= R \sqrt{\frac{C}{L}} = \text{constant} \\
\beta &= \omega \sqrt{LC} \\
u_p &= \omega / \beta = 1/\sqrt{LC} = \text{constant} \Rightarrow \text{no distortion} \\
Z_0 &= \cdots = \sqrt{L/C} \\
\Rightarrow R_0 &= \sqrt{L/C}, X_0 = 0
\end{align*}
\]

Ex 9-3: 50Ω Distortionless lines
9.4 Wave on Finite Transmission Lines

1. For a general transmission line,
\[
\begin{align*}
V(z) &= V_0^+e^{-\gamma z} + V_0^-e^{+\gamma z} \\
I(z) &= I_0^+e^{-\gamma z} + I_0^-e^{+\gamma z} \\
V_0^+ &= -\frac{V_0^-}{Z_0} = Z_0 = \text{characteristic impedance}
\end{align*}
\]

2. Now consider a finite line \(Z_0\) terminated with a load impedance \(Z_L\), at \(z = \ell\), \(V\) and \(I\) must satisfy

(a) If \(Z_L = Z_0\) (matched line) \(\Rightarrow\) no backward (reflected) wave

(b) If \(Z_L \neq Z_0\) (unmatched line) \(\Rightarrow\) a reflected wave must exit in order to satisfy the above boundary condition

3. \(V(z)\) and \(I(z)\) can be expressed in terms of \(V_i, I_i\) or \(V_L, I_L\):
\[
\begin{align*}
V(\ell) &= V_L = V_0^+e^{-\gamma \ell} + V_0^-e^{+\gamma \ell} \\
I(\ell) &= I_L = \frac{V_0^+}{Z_0}e^{-\gamma \ell} - \frac{V_0^-}{Z_0}e^{+\gamma \ell} \\
&\Rightarrow \begin{cases} 
V_0^+ = \frac{1}{2}(V_L + I_L Z_0)e^{\gamma \ell} \\
V_0^- = \frac{1}{2}(V_L - I_L Z_0)e^{-\gamma \ell}
\end{cases}
\end{align*}
\]

Let \(z' = \ell - z\) = distance backward from the load
\[
\begin{align*}
V(z') &= \frac{I_L}{2Z_0}[(Z_L + Z_0)e^{\gamma z'} + (Z_L - Z_0)e^{-\gamma z'}] \\
I(z') &= \frac{I_L}{2Z_0}[(Z_L + Z_0)e^{\gamma z'} - (Z_L - Z_0)e^{-\gamma z'}]
\end{align*}
\]

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Or, \( V(z') = \frac{I}{2}[Z_L(e^{\gamma z'} + e^{-\gamma z'}) + Z_0(e^{\gamma z'} - e^{-\gamma z'})] \)

\[\Rightarrow\]

\[\Rightarrow Z(z') = \frac{V(z')}{I(z')} = \text{impedance at } z' \text{ looking toward the load} \]

4. At the source, \( z' = \ell, \)

\( Z_i = \text{input impedance seen by the generator} \)

\[\Rightarrow \text{equivalent circuit} \]

5. Average power delivered by the generator to the line:

\[ (P_{av})_i = \frac{1}{2} \Re[V_i I_i^*]|_{z=0, z' = \ell} \]

Average power delivered to the load

\[ (P_{av})_L = \frac{1}{2} \Re[V_L I_L^*]|_{z=\ell, z' = 0} = \frac{1}{2} \left| \frac{V_L}{Z_L} \right|^2 R_L = \frac{1}{2} I_L^2 R_L \]

(for lossless lines, \( (P_{av})_i = (P_{av})_L \))

6. If \( Z_L = Z_0 \) (matched),

\[ \rightarrow (9-98) \text{ is reduced to} \]

\[
\begin{cases}
V(z) &= V_i e^{-\gamma z} \\
I(z) &= V_i e^{-\gamma z}
\end{cases}
\]

\[\Rightarrow \text{no reflection wave, similar to infinite line} \]
Ex 9-5: 50Ω lossless matched line

(a) instantaneous $V$ and $I$ at arbitrary locations

(b) instantaneous $V$ and $I$ at load

(c) average power to the load

9.4.1 Transmission Lines as Circuit Elements

1. At UHF, $f = 300$ MHz $\sim 3$ GHz, $\lambda=1$ m $\sim 0.1$ m. Lumped elements are difficult to make (stray field effect).
   ⇒ finite transmission line can be used as circuit elements.

2. Consider a lossless transmission line segment,

\[
Z(z') = Z_0 \frac{Z_L + Z_0 \tanh \gamma z'}{Z_0 + Z_L \tanh \gamma z'}
\]

©Yi Chiu, ECE Dept., NCTU 9-7
(a) $Z_L \to \infty$ (open circuit termination)

(Note: It is difficult to obtain $Z_L \to \infty$ due to field coupling and radiation.)

(b) $Z_L = 0$ (short circuit termination)

(c) Quarter wave section ($\ell = \lambda/4, \beta\ell = \pi/2$)
   
or $\ell = (2n - 1)\lambda/4, n = 1, 2, 3..., \Rightarrow \tan \beta\ell = \pm\infty$
   
   $\Rightarrow$

(d) Half wave section ($\ell = \lambda/2, \beta\ell = \pi$)
   
or $\ell = n\lambda/2, n = 1, 2, 3..., \Rightarrow \tan \beta\ell = 0$
   
   $\Rightarrow Z_i = Z_L$ (why?)

3. By measuring $Z_{io} = Z_0 \coth \gamma\ell$ and $Z_{is} = Z_0 \tanh \gamma\ell$, we can determine

$Z_0 = \sqrt{Z_{io} Z_{is}} \quad (\Omega)$

$\gamma = \frac{1}{\ell} \tanh^{-1} \sqrt{\frac{Z_{is}}{Z_{io}}} \quad (\text{m}^{-1})$
Ex 9-6: Open- and short-circuited impedance

(a)

(b)

(c)

9.4.2 Resistive Termination

1. For $z' = \ell - z$,

$$V(z') = \frac{I_L}{2} \left[ (Z_L + Z_0)e^{\gamma z'} + (Z_L - Z_0)e^{-\gamma z'} \right]$$

\[\text{\(-z'\) incident wave} \quad \text{\(+z'\) reflected wave}\]

Similarly,

$$I(z') = \frac{I_L}{2Z_0}(Z_L + Z_0)e^{\gamma z'}(1 - \Gamma e^{-2\gamma z'})$$
2. For lossless lines, $\gamma = j\beta, Z_0 = R_0$,

\[
\begin{align*}
V(z') &= \frac{I_L}{Z_L}(Z_L + R_0)e^{j\beta z'}(1 + |\Gamma|e^{j(\theta_r - 2\beta z')}) \\
I(z') &= \frac{I_L}{R_0}(Z_L + R_0)e^{j\beta z'}(1 - |\Gamma|e^{j(\theta_r - 2\beta z')})
\end{align*}
\tag{9-135}
\]

Or from (9-100, p. 9-6) and $V_L = I_L Z_L, Z_L = R_L$ (purely resistive load)

\[
\begin{align*}
|V(z')| &= V_L \sqrt{\cos^2 \beta z' + (R_0/R_L)^2 \sin^2 \beta z'} \\
|I(z')| &= I_L \sqrt{\cos^2 \beta z' + (R_0/R_L)^2 \sin^2 \beta z'}
\end{align*}
\tag{9-137}
\]

3. Sanding wave on transmission lines

(a) For $R_L = R_0$, or, in general, $Z_L = Z_0$, there is no reflected wave ($\Gamma = 0$) and $|V(z')| = V_L = \text{constant}$

(b) For $R_L \neq R_0$, or, in general, $Z_L \neq Z_0$, there is a reflected wave ($\Gamma \neq 0$) and $|V(z')| \neq \text{constant}$ due to the interference of the two waves

(Note: the period of the standing wave pattern is $\lambda/2$)

(c) The standing wave ratio is defined as

For a lossless line,

- $Z_L = Z_0$ (matched) $\Rightarrow \Gamma = 0, S = 1$
- $Z_L = 0$ (short circuited) $\Rightarrow \Gamma = -1, S \rightarrow \infty$
- $Z_L = \infty$ (open circuited) $\Rightarrow \Gamma = 1, S \rightarrow \infty$

4. max and min of $V$ and $I$ on the transmission line for general load

From (9-135),

$|V_{\text{max}}|, |I_{\text{min}}|$ occurs at $\theta_r - 2\beta z' = -2n\pi, n = 0, 1, 2, ...$

$|V_{\text{min}}|, |I_{\text{max}}|$ occurs at $\theta_r - 2\beta z' = -(2n + 1)\pi, n = 1, 2, ...$
For resistive termination on a lossless line, both $Z_L$ and $Z_0$ are real,

(1) $R_L > R_0, \theta_\Gamma = 0$

(2) $R_L < R_0, \theta_\Gamma = \pi$

9.4.3 Arbitrary Termination (see Smith Chart)

9.4.4 Transmission Line Circuit

1. To express $V(z')$ in terms of $V_g$, let us consider the multiple reflections at $z = 0$ and $z = \ell$:

$V(z') = \sum$ all waves traveling in the line

$= V_1^+ + V_1^- + V_2^+ + V_2^- + ...$

If $Z_L = Z_0 \Rightarrow$ only $V_1^+$ exists ($\Gamma = 0$)

If $Z_L \neq Z_0$ but $Z_g = Z_0 \rightarrow \Gamma \neq 0, \Gamma_g = 0 \Rightarrow$ only $V_1^+$ and $V_1^-$ exist
Ex 9-10: 50 Ω lossless air line
9.6 Smith Chart

1. For a lossless line, the reflection coefficient $\Gamma$ is

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{Z_L - R_0}{Z_L + R_0} = |\Gamma|e^{j\theta}$$

2. If the load impedance is normalized to the line impedance $R_0$,

$$r = 1 - \frac{\Gamma^2}{1 - \Gamma^2}$$

$$x = \frac{2\Gamma}{(1 - \Gamma^2)^2 + \Gamma^2}$$

3. The real part of the normalized load impedance $r$ can be plotted in the complex $\Gamma$ plane. It can be shown that

$$\Rightarrow$$ circle centered at $(\Gamma_r, \Gamma_i) = \left(\frac{r}{1 + r}, 0\right)$ with radius $\frac{1}{1 + r}$

Similarly, the imaginary part $x$ satisfies

$$\Rightarrow$$ circle centered at $(1, \frac{1}{x})$ with radius $\frac{1}{x}$
(a) Constant-\(r\) circles (Eq. 9-188):
   i. centered on the \(\Gamma_r\) axis (\(\Gamma_i = 0\))
   ii. largest circle: \(r = 0\) centered at \((0, 0)\)
   iii. for \(r \to \infty\), \(r\)-circle \(\to (1, 0)\) (open circuit)
   iv. all \(r\)-circles pass through \((1, 0)\)

(b) Constant-\(x\) circles (Eq. 9-189):
   i. centered on the \(\Gamma_r = 1\) line
   ii. for \(x = 0\) \(\Rightarrow \Gamma_r\) axis
   iii. for \(x = 0 \to \infty\), \(x\)-circle \(\to (1, 0)\) (open circuit)
   iv. all \(x\)-circles pass through \((1, 0)\)

4. Smith Chart:

   (a) \(r\)- and \(x\)-circles are plotted in \(\Gamma_r - \Gamma_i\) plane for \(|\Gamma| \leq 1\);

   (b) Intersection of a \(r\)- and a \(x\)-circle represents a normalized load
       impedance \(z_L = r + jx\).

Ex: In Fig. 9-30, the point \(P\) is found to be \((r, x) = (1.7, 0.6)\). Therefore, the normalized impedance is

and the real load impedance is

Some interesting points on the chart:
\(P_{SC} : (r, x) = \) \\
\(P_{OC} : (r, x) = \)

(c) Once the point representing the normalized impedance is located, 
   the reflection coefficient \(\Gamma = |\Gamma|e^{i\theta_r}\) can be found from the chart, 
   and vise versa.
The Complete Smith Chart
Black Magic Design

(Source: Wikipedia)
(d) Constant-$|\Gamma|$ circles:
   i. centered at (0, 0) and $0 \leq |\text{radius}| \leq 1$
   ii. $P_M, P_m$:

$\Rightarrow \text{SWR} = \text{the } r\text{-circle passing through } P_M$

5. Input impedance calculation using Smith Chart

$$Z_i(z') = \frac{V(z')}{I(z')} = Z_0 \frac{1 + \Gamma e^{-j2\beta z'}}{1 - \Gamma e^{-j2\beta z'}} \quad (\text{Eq. 9-133, } \gamma = j\beta)$$

$\Rightarrow$ on the $\Gamma$-plane (Smith chart), $z_i(z')$ is located on the $|\Gamma|$-circle with phase angle $\phi = \theta_\Gamma - 2\beta'$

But $|\Gamma|$ and $S$ are independent of $z'$

$\Rightarrow z_i(z')$ can be independent from proper rotation of $z_L$ on the $|\Gamma|$-circle:

Given $z_L = z_i(0)$,

(a) find $\Gamma = |\Gamma|e^{j\theta_r}$;

(b) rotate the $\Gamma$ point clockwise along the $|\Gamma|$-circle by an angle of $2\beta z' = 4\pi z'/\lambda$ to $\Gamma'$ (subtraction of phase)

(c) $\Gamma' = |\Gamma|e^{j(\theta_r-2\beta z')} = |\Gamma|e^{j\phi}$ corresponds to the input impedance $z_i$ at $z'$

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The Complete Smith Chart
Black Magic Design

(Source: Wikipedia)
Ex 9-13:
Ex 9-14:
Ex 9-15: Solving Ex 9-9 using Smith Chart
6. Normalized admittance (p. 499)

Smith Chart relates $\Gamma$ and normalized load impedance $z_L = Z_L/Z_0$.

Now consider the load impedance $Y_L$. The normalized load impedance can be defined as

On Smith Chart,

- $z_L$ and $y_L$ are opposite to each other on the $|\Gamma|$-circle.
- Same Smith Chart can be used to relate $\Gamma$ and $z_L$, or $\Gamma'$ and $y_L$.
- $z_L = r + jx \Leftrightarrow y_L = g + jb$

Ex 9-18: $Z_L = 95 + j20 \, \Omega$. Find $Y_L$. 
Ex 9-19:

The Complete Smith Chart
Black Magic Design

(Source: Wikipedia)
9.7 Transmission Line Impedance Matching

1. reduce reflection and loss
2. improve power transmission efficiency
3. reduce signal interference
4. protect signal generator

9.7.1 Quarter wave transformer

- Frequency sensitive (quarter "wavelength")
- If $R_0, R'_0 = \text{real}$, $\Rightarrow R_L$ must also be real $\Rightarrow$ can not be used to match complex load impedance $Z_L$

Ex 9-17:

(a) For equal power to $R_1$ and $R_2$,

(b) Standing wave ratio:
9.7.2 Single stub matching (Stub tuner)

Find ℓ and d such that

In terms of normalized impedance,

But $y_s = g + jb = jb$,

⇒

⇒ Find d such that

and ℓ such that

Ex 9-20:
The Complete Smith Chart
Black Magic Design

(Source: Wikipedia)
The Complete Smith Chart
Black Magic Design

(Source: Wikipedia)