Topic-2 Basic Concepts in Mathematics

2-1 Function

To learn “Calculus”, it is required to know “Function”, which will be briefly introduced in this section.

A. Definition of Function

A function is a rule that maps elements from the domain (or domain of definition) to its range (or co-domain). Since all the functions to be discussed in this course are concerning practical systems, here only consider real numbers as the elements in the domain or range. In general, a function \( f \) is given as

\[
y = f(x)
\]

(2.1)

where \( x \in \mathbb{R} \) is in the domain, \( y \in \mathbb{R} \) is in the range, and \( f \) is a symbol to express their relationship as shown in Fig.2.1. Clearly, \( y \) is changed while \( x \) changes. In other words, \( y \) is a dependent variable whose value varies according to the independent variable \( x \). Note that (2.1) is only a function that maps a single variable to the other single variable. Sometimes, a function may map multiple variables to a single variable or multiple variables. For example, consider a helicopter hovering at \((x,y,z)\) where \( z \) is the height above the location \((x,y)\) on the horizontal plane. Then, its height \( z \) can be expressed as a function of \((x,y)\), i.e., \( z = f(x,y) \). That’s a mapping from multiple variables to a single variable. For the mapping between multiple variables, it is easy to take the coordinate transformation as an example. For instance, we have learned that \( x = r \cos \theta \) and \( y = r \sin \theta \), a transformation between Cartesian coordinate and Polar coordinate. This kind of mapping is commonly expressed as a vector form, which will be discussed in the course “Linear Algebra.”

\[\text{Fig.2.1}\]

B. Types

The types of function in real system are often classified into time-dependent, space-dependent, and time-space-dependent functions. Let \( t \) and \( x \) be the independent variables, representing time and position respectively. A time-dependent function is, for example, given as
\[ f(t) = \frac{1}{2}at^2 + bt + c \quad \text{for} \quad t_0 \leq t \leq t_1 \]  

(2.2)

where \( t \in [t_0, t_1] \) and the coefficients \( a, b \) and \( c \) are constant. It is known as a function to represent an moving object with constant acceleration \( a \), initial velocity \( b \) and initial position \( c \).

For a space-dependent function, let's consider the Coulomb’s law as an example, which is expressed as the following form

\[ F_e(x) = \frac{kQq}{x^2} \quad \text{for} \quad 0 < x < \infty \]  

(2.3)

where \( F_e(x) \) represents the electrostatic force between two charges \( Q \) and \( q \) separated by a distance \( x \).

When a function depends not only on time variable \( t \) but also on space variable \( x \), it is called a time-space-dependent function. For example, the wave function traveling in space is given as

\[ E(x, t) = A \cdot \sin \left( \frac{2\pi x}{\lambda} - 2\pi ft \right) \quad \text{for} \quad 0 \leq x < \infty, \quad \text{and} \quad 0 \leq t < \infty \]  

(2.4)

where \( x \in [0, \infty) \) and \( t \in [0, \infty) \). Note that \( E(x,t) \) represents the electric intensity of the electromagnetic wave propagating in the space with magnitude \( A \), wavelength \( \lambda \) and frequency \( f \).

2-2 Calculus

A. Differentiation

The differentiation is the most fundamental operation to analyze the change of a continuous function \( y(x) \) with respect to \( x \). Consider the curve \( y(x) \) in Fig.2.2, which is continuous at \( x=x_0 \). The slope \( m_i \) between \( (x_0, y(x_0)) \) and \( (x_i, y(x_i)) \) on the curve is given as

\[ m_i = \frac{y(x_i) - y(x_0)}{x_i - x_0}, \quad i=1,2,3,\ldots \]  

(2.5)

where the difference \( x_i-x_0 \) is getting smaller as \( i \) increases. Evidently, as \( i \) approaches infinite, i.e.,
As $i \to \infty$, the slope $m_i$ is defined as the slope of $y(x)$ at $x=x_0$ and expressed as

$$m_i = \lim_{x \to x_0} \frac{y(x) - y(x_0)}{x - x_0}$$

(2.6)

Mathematically, $m_i$ is also denoted as $\dot{y}(x_0)$, $y'(x_0)$ or $\frac{dy(x)}{dx} \bigg|_{x=x_0}$. Then, by letting $x=x_0+\Delta x$, (2.6) can be rewritten as

$$\dot{y}(x_0) = \lim_{\Delta x \to 0} \frac{y(x_0 + \Delta x) - y(x_0)}{\Delta x}$$

(2.7)

which is a form usually used to represent the differentiation at a specified $x_0$. If $y(x)$ is continuous for all $x$, (2.7) can be changed into

$$\dot{y}(x) = \lim_{\Delta x \to 0} \frac{y(x + \Delta x) - y(x)}{\Delta x}$$

(2.8)

Obviously, $\dot{y}(x)$ is also a function and formally called the derivative of $y(x)$.

In fact, the differentiation can be also applied to the function $\dot{y}(x)$ when it is continuous. Such differentiation is denoted as $\ddot{y}(x)$, $y''(x)$ or $\frac{d^2y(x)}{dx^2}$, called the second derivative of $y(x)$. Following the same way, we can have the third derivative of $y(x)$, denoted as $\dddot{y}(x)$, $y'''(x)$ or $\frac{d^3y(x)}{dx^3}$. For the $n$th derivatives, $n>3$, it is convenient to use the form $\frac{d^n y(x)}{dx^n}$ or $y^{(n)}(x)$.

In engineering, the derivative is often used for “prediction”, i.e., to predict the future of a time-dependent function. Consider the function $f(t)$ in Fig.2.3. Once the slope of $f(t)$ at $t=t_0$ is known, i.e., $f'(t_0)$ is obtained, the value of $f(t_0+\Delta t)$ can be predicted as

$$\dot{f}(t_0 + \Delta t) = f(t_0) + f'(t_0) \Delta t$$

(2.9)

with prediction error.
\[ \varepsilon (t_0) = f (t_0 + \Delta t) - \hat{f} (t_0 + \Delta t) \]  
(2.10)

Clearly, the smaller the time duration \( \Delta t \), the smaller the prediction error \( \varepsilon (t_0) \).

Finally, let’s introduce the most important function, natural exponential function, which is expressed as

\[
e^x = \lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^n = \lim_{n \to \infty} \left( 1 + C_1^n \frac{x}{n} + C_2^n \frac{x^2}{n^2} + C_3^n \frac{x^3}{n^3} + \cdots \right)
\]

\[
= \lim_{n \to \infty} \left( 1 + \frac{x}{1!} \frac{x}{n} + \frac{n(n-1)}{2!} \frac{x^2}{n^2} + \frac{n(n-1)(n-2)}{3!} \frac{x^3}{n^3} + \cdots \right) 
\]

\[= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \]  
(2.11)

which is a very useful function since almost all the natural phenomena can be described by \( e^x \). It is also the reason why \( e^x \) is named as “natural”. Most significantly, the derivative of \( e^x \) is itself, i.e.,

\[
\frac{d e^x}{dx} = 0 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x 
\]
(2.12)

Now, one question is also raised: Why \( \sum_{n=0}^{\infty} \frac{x^n}{n!} \) is an exponential function? In other words, how can we know \( e^x e^y = e^{x+y} \)? Actually, it can be proved by direct calculation:

\[
e^x e^y = \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{m=0}^{\infty} \frac{y^m}{m!} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{x^n y^m}{n! m!}
\]

\[
= \sum_{k=0}^{\infty} \sum_{p=0}^{k} \frac{k!}{p!} \frac{x^p y^{k-p}}{(k-p)!} p! (k-p)!
\]

\[= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{p=0}^{k} C_k^p x^p y^{k-p} \]

\[= \sum_{k=0}^{\infty} \frac{1}{k!} (x+y)^k = e^{x+y} 
\]  
(2.13)

Clearly, \( e^x \) is indeed an exponential function. By letting \( x=1 \) and \( x=-1 \) in (2.11), we have

\[
e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = 2.718 \cdots 
\]

and \( e^{-1} = 0.3679 \cdots \)  
(2.14)

Both are important constants in the engineering.

**Example 2.1**

Find the derivative of \( y(x) = \frac{1}{x+1} \) with respect to \( x \).
Sol:

According to (2.8), we have

$$\dot{y}(x) = \lim_{\Delta x \to 0} \frac{y(x+\Delta x) - y(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left(\frac{1}{(x+\Delta x)+1} - \frac{1}{x+1}\right)$$

$$= \lim_{\Delta x \to 0} \frac{-\Delta x}{\Delta x(x+\Delta x+1)(x+1)}$$

$$= \lim_{\Delta x \to 0} \frac{-1}{(x+\Delta x+1)(x+1)} = \frac{-1}{(x+1)^2}$$

**Example 2.2**

Find the derivative of  \( f(t) = \sin 2t \) with respect to  \( t \).

**Sol:**

According to (2.8), we have

$$\dot{f}(t) = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\sin(2(t+\Delta t)) - \sin 2t}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(\sin(2t)\cos(2\Delta t) + \cos(2t)\sin(2\Delta t) - \sin 2t\right)$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(\sin(2t) + \cos(2t)\cdot(2\Delta t) - \sin 2t\right)$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(\cos(2t)\cdot(2\Delta t)\right) = 2\cos 2t$$

Here, we adopt the truth of  \( \sin \Delta t \approx \Delta t \) and  \( \cos \Delta t \approx 1 \) for  \( \Delta t \approx 0 \).

**[Problem 2.1]**

Find the derivatives of the following functions:

(A)  \( \cos(t) \)  
(B)  \( \frac{1}{t^2+1} \)  
(C)  \( e^{at} \)  
(D)  \( \tan(5t) \)  
(E)  \( \frac{t}{t^2+1} \)

**B. Integration**

The integration is the reverse operation of differentiation and accumulates the values of  \( y(x) \) from  \( x = x_1 \) to  \( x = x_2 \), denoted as

$$I = \int_{x_1}^{x_2} y(x) \, dx \quad (2.15)$$
which is the integral from $x_1$ to $x_2$ with respect to $x$. Note that $y(x)$ can be piecewise continuous or discontinuous as shown in Fig.2.4.

The integral of $y(x)$ is equal to the area bounded between the curve $y(x)$ and the $x$-axis, where the area can be positive or negative. If $x_2>x_1$ and $y(x)>0$, then the result of (2.11) is positive, For example the integral $I_1$ in Fig.2.4 is

$$I_1 = \int_{x_1}^{x_2} y(x) \, dx > 0$$

If $x_2>x_1$ and $y(x)<0$, then the result of (2.11) is negative, for example

$$I_3 = \int_{x_1}^{x_2} y(x) \, dx < 0$$

If $x_2>x_1$ and $y(x)$ changes its numeric sign within $x \in (x_1, x_2)$, then the integral may be positive or negative, for example

$$\int_{x_1}^{x_2} y(x) \, dx = I_2 + I_3 = \int_{x_1}^{x_3} y(x) \, dx + \int_{x_3}^{x_2} y(x) \, dx$$  \hspace{1cm} (2.18)

where $I_2>0$ and $I_3<0$. Whether $\int_{x_1}^{x_2} y(x) \, dx$ is positive or negative depends on the result of $I_2 + I_3$.

In some situations, integration may be taken in a different direction, such as from $x=x_2$ to $x=x_1$. Mathematically, it is defined as

$$\int_{x_2}^{x_1} y(x) \, dx = -\int_{x_1}^{x_2} y(x) \, dx$$  \hspace{1cm} (2.19)

Besides, integration can be taken part by part, for example

$$\int_{x_1}^{x_2} y(x) \, dx = \int_{x_1}^{x_5} y(x) \, dx + \int_{x_5}^{x_2} y(x) \, dx$$  \hspace{1cm} (2.20)

just like the way shown in (2.18). Moreover, it is noticed that $x$ in (2.15) is a dummy variable, which means the integral $I$ will not change even though $x$ is replaced by a different symbol. For example, the following integrals are the same:

$$\int_{x_1}^{x_2} y(x) \, dx = \int_{x_1}^{x_2} y(t) \, dt = \int_{x_1}^{x_2} y(\tau) \, d\tau$$  \hspace{1cm} (2.21)

The concept of dummy variable may be tedious, but it is somewhat important if you want to clearly express an integral without any confusion.
Consider a time function \( f(t) \). Sometimes, we need the integration of \( f(t) \) measured from a specified time \( t=t_0 \) to another specified time \( t=t_1 \). Then, the result is

\[
F = \int_{t_0}^{t_1} f(t) \, dt
\]  
(2.22)

When we deal with a real-time problem, the integration is usually operated from a specified time \( t=t_0 \) up to the present time \( t \), and then keeps going to the future. To express such integration, a form of “running integral” is always given and expressed as

\[
F(t) = \int_{t_0}^{t} f(\tau) \, d\tau
\]  
(2.23)

where \( F(t) \) is taken by “running with time” and thus it is a function of time \( t \). Here, in order not to make any confusion, we replace the integrand \( f(t) \) in (2.22) by \( f(\tau) \), in terms of a dummy variable \( \tau \).

Now, one important concept should be declared here. Since \( F \) in (2.22) is a constant, its derivative with respect to \( t \) is zero, i.e.,

\[
\frac{dF}{dt} = \frac{d}{dt} \int_{t_0}^{t} f(t) \, dt = 0
\]  
(2.24)

However, the derivative of \( F(t) \) in (2.23) is expressed as

\[
\frac{dF(t)}{dt} = \frac{d}{dt} \int_{t_0}^{t} f(\tau) \, d\tau = f(t)
\]  
(2.25)

which means the integrand is the derivative of a running integral. Let’s derive (2.25) by the definition of derivative:

\[
\frac{dF(t)}{dt} = \lim_{\Delta t \to 0} \frac{F(t+\Delta t)-F(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left( \int_{t_0}^{t+\Delta t} f(\tau) \, d\tau - \int_{t_0}^{t} f(\tau) \, d\tau \right)
\]  
(2.26)

Since

\[
\int_{t}^{t+\Delta t} f(\tau) \, d\tau = \int_{t_0}^{t+\Delta t} f(\tau) \, d\tau - \int_{t_0}^{t} f(\tau) \, d\tau
\]  
(2.27)

the derivative in (2.26) can be rearranged as

\[
\frac{dF(t)}{dt} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{t}^{t+\Delta t} f(\tau) \, d\tau
\]  
(2.28)

Note that \( F(t) = \int_{t_0}^{t} f(\tau) \, d\tau \) is the shadowed area in Fig.2.5 and \( \int_{t}^{t+\Delta t} f(\tau) \, d\tau \) is the slashed area. As \( \Delta t \to 0 \), the slashed area can be treated as a rectangle with area \( \Delta t \cdot f(t) \). Then, (2.28) is changed into
\[
\frac{dF(t)}{dt} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{t}^{t+\Delta t} f(\tau) \, d\tau = \lim_{\Delta t \to 0} \frac{\Delta t \cdot f(t)}{\Delta t} = f(t)
\]  
(2.29)

which proves (2.25).

From (2.25), it is known that \( f(t) \) is the derivative of \( F(t) \). In an inverse sense, \( F(t) \) is “an” inverse derivative of \( f(t) \). Mathematically, we call \( F(t) \) “an” antiderivative of \( f(t) \), not “the” antiderivative of \( f(t) \). In other words, \( F(t) \) is not unique because \( t_0 \) in (2.25) is arbitrary. For simplicity, the antiderivative of \( f(t) \) is often expressed as \( F(t) = \int f(t) \, dt \), not in the form of running integration. In addition, the antiderivative \( F(t) = \int f(t) \, dt \) is also a function of \( t \), just like the derivative.

From \( F(t) = \int f(\tau) \, d\tau \), we have \( F(t_1) = \int_{t_0}^{t_1} f(\tau) \, d\tau \) and \( F(t_2) = \int_{t_0}^{t_2} f(\tau) \, d\tau \). Now, the integral of \( f(t) \) from \( t_1 \) to \( t_2 \) can be expressed by its antiderivative \( F(t) \) as below:

\[
\int_{t_0}^{t_1} f(\tau) \, d\tau + \int_{t_1}^{t_2} f(\tau) \, d\tau = \int_{t_0}^{t_2} f(\tau) \, d\tau = F(t_2) - F(t_1)
\]

(2.30)

Note that both (2.25) and (2.30), i.e.,

\[
\frac{dF(t)}{dt} = \frac{d}{dt} \int_{t_0}^{t} f(\tau) \, d\tau = f(t)
\]

(2.31)

\[
\int_{t_0}^{t_2} f(\tau) \, d\tau = F(t)|_{t_0}^{t_2} = F(t_2) - F(t_1)
\]

(2.32)

are the most fundamental operations in calculus.

Usually, it is not an easy work to find an antiderivative of a function. In order to make the integration easier, a lot of special antiderivatives have been collected and shown as integral tables. Hence, when we are dealing with complicated integration, we should know how to check the integral tables.

**Example 2.3**

Find the antiderivative of \( f(t) = t - t^4 \) and determine its integral from \( t=1 \) to \( t=3 \).

**Sol:**

Since \( \frac{dt^{n+1}}{dt} = (n+1)t^n \), the antiderivative of \( t^n \) is \( \frac{t^{n+1}}{n+1} + c \) where \( c \) is a fixed number and can be given arbitrarily. Hence, the antiderivative of \( f(t) \) is
\[ F(t) = \int f(t) \, dt = \frac{1}{2} t^2 - \frac{1}{5} t^5 + c \]

From (2.32), the integral of \( f(t) \) from \( t=1 \) to \( t=3 \) is

\[ \left[ \int_1^3 f(\tau) \, d\tau \right] = F(3) - F(1) = \left( \frac{9}{2} - \frac{243}{5} + c \right) - \left( \frac{1}{2} - \frac{1}{5} + c \right) = -\frac{222}{5} \]

**Example 2.4**

Find the antiderivative of \( z(x) = e^{2x} + 4 \cos \left( \frac{x}{2} \right) \) and determine its integral from \( x=-1 \) to \( x=1 \).

**Sol:**

Since \( \frac{de^{ax}}{dx} = ae^{ax} \) and \( \frac{d\sin(bx)}{dx} = b \cdot \cos(bx) \), the antiderivatives of \( e^{ax} \) and \( \cos(bx) \) are

\[ \frac{1}{a} e^{ax} + c \quad \text{and} \quad \frac{1}{b} \sin(bx) + c \], where \( c \) is an arbitrary fixed number. Thus, the antiderivative of \( z(x) \) is

\[ z(x) = \int z(x) \, dx = \frac{1}{2} e^{2x} + 8 \sin \left( \frac{x}{2} \right) + c \]

From (2.32), the integral of \( z(x) \) from \( t=-1 \) to \( t=1 \) is

\[ \int_{-1}^{1} z(x) \, dx = \left[ \frac{1}{2} e^{2x} + 8 \sin \left( \frac{x}{2} \right) + c \right]_{-1}^{1} = Z(1) - Z(-1) = \left( \frac{1}{2} e^2 + 8 \sin \left( \frac{1}{2} \right) + c \right) - \left( \frac{1}{2} e^{-2} + 8 \sin \left( -\frac{1}{2} \right) + c \right) = \frac{1}{2} (e^2 - e^{-2}) \]

**[Problem 2.2]**

Find the antiderivatives of the following functions and their integrals from \( t=-1 \) to \( t=1 \):  

(A) \( t+0.5t^2-t^3 \)  
(B) \( \frac{1}{(t+2)^2} \)  
(C) \( e^{3t+1} \)  
(D) \( \cos(5t) \)  
(E) \( \sin^2(t) \)