§13.3 Arc Length and Curvature

1. **Arc length** : \( r(t) = \langle x(t), y(t) \rangle \text{ or } \langle x(t), y(t), z(t) \rangle \)

\[
L = \int_{a}^{b} \sqrt{(dx)^2 + (dy)^2} \\
= \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \\
= \int_{a}^{b} |r'(t)| \, dt
\]

(i). 從物理角度看：

- \( r'(t) \) 代表物體在 \( t \) 時間的瞬間速度
- \( |r'(t)| \) 代表物體在 \( t \) 時間的瞬間速率
- \( |r'(t)| \Delta t = \) 小範圍的距離

\[
\Rightarrow \int_{a}^{b} |r'(t)| \, dt = \text{代表此物體從 } t = a \text{ 到 } t = b \text{ 所走過的距離}
\]

2. **Arc length function** ( or Distance Function )

\[
s(t) = \int_{a}^{t} |r'(u)| \, du. \\
\Rightarrow \frac{ds}{dt} = |r'(t)| = \text{距離的變化率} = \text{速度}.
\]

**Example 1**:

Find the length of the curve \( r(t) = \left\langle \sin 2t, \cos 2t, 2t^2 \right\rangle, 0 \leq t \leq 1. \)

\[
(A) \, \frac{2}{27} \left(13\sqrt{13} - 8\right) \quad (B) \, \frac{13}{9} \quad (C) \, \frac{13\sqrt{13} - 6}{27} \quad (D) \, \frac{16}{9}
\]

**Solution** :

(A)

**Example 2**:

Let \( C \) be a curve described by \( x = f(t), y = g(t), \alpha \leq t \leq \beta, \) where \( f' \) and \( g' \) are continuous on [ \( \alpha, \beta \) ] and \( C \) is traversed exactly once as \( t \) runs from \( \alpha \) to \( \beta \). Which one of the following is always true?
(A) \[ \int_{\alpha}^{\beta} \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2} \, dt \geq \beta - \alpha \]

(B) \[ \int_{\alpha}^{\beta} \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2} \, dt \geq \sqrt{\beta^2 + \alpha^2} \]

(C) \[ \int_{\alpha}^{\beta} \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2} \, dt \geq \beta - \alpha \]

(D) \[ \int_{\alpha}^{\beta} \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 \, dt \geq \sqrt{(y(\beta) - y(\alpha))^2 + (x(\beta) - x(\alpha))^2} \]

Solution :

(D)

Example 3 :

Let the distance traveled by a particle with position

\((x(t), y(t)) = (\sin^2 t, \cos^2 t)\) as \(t\) varies from \(t = 0\) to \(t = 3\pi\) be \(d\).

Then \(d = ?\)

(A) \(\sqrt{2}\)  (B) 0  (C) 6\(\sqrt{2}\)  (D) 4\(\sqrt{2}\)

Solution :

(C)

3. 將一個 curve 用 arc length(s) 來作參數式是一個非常有用的想法和技巧。
(如此的表達方式不隨著不同座標系統而改變)

4. Unit tangent vector : \(T(t) = \frac{r'(t)}{|r'(t)|}\).
5. **Curvature** (曲率) : a measure of how quickly the curve changes direction at a given point.

**Definition** :
\[ \kappa = \frac{dT}{ds} \]

**Example 4** :
Show that the curvature of a circle with radius \( a \) is \( \frac{1}{a} \).

**Theorem** :
(i). \[ \kappa = \frac{dT}{ds} = \frac{dT}{ds} = \left| \frac{T'(t)}{r'(t)} \right| = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} \]

(ii). Given a plane curve \( y = f(x) \), then its curvature \( \kappa \) at a given point \( x \) is
\[ \kappa(x) = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}} \]

**Proof** :
(i). \[ r' = \left| r' \right| T = \frac{ds}{dt} T \quad \text{(1)} \]

\[ r'' = \frac{d^2 s}{dt^2} T + \frac{ds}{dt} T' \quad \text{(2)} \]

\[ (1) \times (2) \]

\[ r' \times r'' = \left( \frac{ds}{dt} \right)^2 (T \times T'). \]

\[ |r' \times r''| = \left( \frac{ds}{dt} \right)^2 |T||T'| = \left( \frac{ds}{dt} \right)^2 |T'| \]

\[ |T'| = \frac{|r' \times r''|}{\left( \frac{ds}{dt} \right)^2} = \frac{|r' \times r''|}{|r'|^2} \]

\[ \kappa = \frac{|T'|}{|r'|} = \frac{|r' \times r''|}{|r'|^3} \]

6. **Principal unit normal vector** \( N(t) \).
\[ N(t) = \frac{T'(t)}{|T'(t)|} \]
7. Binormal vector $B(t)$.

\[ B(t) = T(t) \times N(t) \]