A multi-band approach to arterial traffic signal optimization

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MILP-1
The basic, symmetric, uniform-width bandwidth maximization problem

MILP-2
- Extends the basic problem to include asymmetric bandwidths in opposing directions, variable left-turn phase sequence, as well as decisions on cycle time length and link specific progression speeds

MILP-3
- Presents the new multi-band, multi-weight approach, which also incorporates all previous decision capabilities
$w_i + b \leq 1 - r_i \quad (1a)$

Interference variable

Queue clearance time

Outbound bandwidth

Red time (Inbound)

Red time (outbound)
Inbound

Red time (Inbound)

\[ w_i + b \leq 1 - r_i \] (1b)

Interference variable

Inbound bandwidth

Queue clearance time
Inbound

Internode offsets

$\phi (3, 4)$
\[ \phi(1,4) + \phi(1,4) + \Delta_1 - \Delta_4 = m(1,4) \quad (2) \]

\( m \) is an integer variable

Intranode offset of intersection 1
Outbound

\[ \phi (1,4) = 0.5 \times r_1 \]  

(3a)

Travel time from S1 to S4 outbound
\[
\phi(1,4) + 0.5 \cdot \bar{r}_4 + \bar{w}_4 = 0.5 \cdot \bar{r}_1 + \bar{w}_1 - \bar{\tau}_1 + \bar{t}(1,4) \tag{3b}
\]
\( \phi(1,4) + \bar{\phi}(1,4) + \Delta_4 - \Delta_1 = m(1,4) \) (2)

\[ \phi(1,4) + 0.5 \cdot r_4 + w_4 + \tau_4 = 0.5 \cdot r_1 + w_1 + t(1,4) \] (3a)

\[ \bar{\phi}(1,4) + 0.5 \cdot \bar{r}_4 + \bar{w}_4 = 0.5 \cdot \bar{r}_1 + \bar{w}_1 - \tau_1 + \bar{t}(1,4) \] (3b)

Substituting (3) into (2):

\[ t(1,4) + \bar{t}(1,4) + 0.5 \cdot (\bar{r}_1 + r_1) + (\bar{w}_1 + w_1) - 0.5 \cdot (\bar{r}_4 + r_4) - (\bar{w}_4 + w_4) - (\tau_4 + \tau_1) + \Delta_1 - \Delta_4 = m(1,4) \]
If we replace intersection 4 by intersection 2:

\[
t(1,2) + \tilde{t}(1,2) + 0.5 \cdot (\bar{r}_1 + r_1) + (\bar{w}_1 + w_1) \\
- 0.5 \cdot (\bar{r}_2 + r_2) - (\bar{w}_2 + w_2) - (\bar{\tau}_2 + \tau_1) + \Delta_1 - \Delta_2 \\
= m(1,2)
\]

\[
t(i, i + 1) + \tilde{t}(i, i + 1) + 0.5 \cdot (\bar{r}_i + r_i) + (\bar{w}_i + w_i) \\
- 0.5 \cdot (\bar{r}_{i+1} + r_{i+1}) - (\bar{w}_{i+1} + w_{i+1}) - (\bar{\tau}_{i+1} + \tau_i) + \Delta_i - \Delta_{i+1} \\
= m(i, i + 1)
\]
We define \( x_i = x(i, i + 1) \)

\[
t_i + t_i + (w_i + w_i) - (w_{i+1} + w_{i+1}) + \Delta_i - \Delta_{i+1} \\
= -0.5 * (r_i + r_i) + 0.5 * (r_{i+1} + r_{i+1}) + (\tau_{i+1} + \tau_i) + m_i
\]
MILP-1

Find $b, w_i, w_i, m_i$ to

$$\text{Max } b = \bar{b}, \text{ subject to}$$

$$w_i + b \leq 1 - r_i$$
$$w_i + b \leq 1 - r_i$$
$$t_i + \frac{1}{2} + (w_i + w_i) - (w_i + w_i) + \Delta_i - \Delta_{i+1}$$

$$= -0.5 \ast (r_i + \bar{r}_i) + 0.5 \ast (r_{i+1} + \bar{r}_{i+1}) + (\tau_{i+1} + \bar{\tau}_i) + m_i$$

$m_i = \text{integer}$

$b, \bar{b}, w_i, \bar{w}_i \geq 0, \quad i = 1, \ldots, n$
- **MILP-1**
  - The basic, symmetric, uniform-width bandwidth maximization problem

- **MILP-2**
  - Extends the basic problem to include asymmetric bandwidths in opposing directions, variable left-turn phase sequence, as well as decisions on cycle time length and link specific progression speeds

- **MILP-3**
  - Presents the new multi-band, multi-weight approach, which also incorporates all previous decision capabilities
MILP-2

- Extension 1
  - The directional weighting of the two bands

\[
\text{Max } (b + kb), \text{ subject to }
\]

\[
\begin{align*}
\bar{b} &\geq kb & \text{if } k < 1 \text{ (outbound favored)} \\
\bar{b} &\leq kb & \text{if } k > 1 \text{ (inbound favored)} \\
\bar{b} &= b & \text{if } k = 1 \text{ (balanced progression)}
\end{align*}
\]

\[
(1 - k)\bar{b} \geq (1 - k)kb
\]
Extension 2

- Let both the common signal cycle time $C$ and the link specific progression speed $v_i$ be optimizable variables.

\[
\frac{1}{C_2} \leq z \leq \frac{1}{C_1} \quad \frac{1}{f_i} \leq \frac{1}{v_i} \leq \frac{1}{e_i} \\
z = \frac{1}{C} \quad \frac{1}{h_i} \leq (\frac{1}{v_{i+1}}) - (\frac{1}{v_i}) \leq \frac{1}{g_i}
\]

$C_1, C_2$ = lower and upper limits on cycle length
$e_i, f_i$ ($\bar{e}_i, \bar{f}_i$) = lower and upper limits on outbound (inbound) speed (feet/second)
$g_i, h_i$ ($\bar{g}_i, \bar{h}_i$) = lower and upper limits on change in outbound (inbound) speed (feet/second)
Distance between $S_h$ and $S_i$  

Progression speed

Travel time (inbound)

Signal frequency

$t_i = \left( \frac{d_i}{v_i} \right) z$

$\bar{t_i} = \left( \frac{\bar{d_i}}{\bar{v_i}} \right) z$
\[
\frac{1}{f_i} \leq \frac{1}{v_i} \leq \frac{1}{e_i} \quad \Rightarrow \quad \left(\frac{d_i}{f_i}\right)z \leq t_i \leq \left(\frac{d_i}{e_i}\right)z
\]

\[
\frac{1}{h_i} \leq \left(\frac{1}{v_{i+1}}\right) - \left(\frac{1}{v_i}\right) \leq \frac{1}{g_i}
\]

\[
\left(\frac{d_i}{h_i}\right)z \leq \left(\frac{d_i}{d_{i+1}}\right)t_{i+1} - t_i \leq \left(\frac{d_i}{g_i}\right)z
\]
- Extension 3
  - Determine the sequence of the left turn phase
    - Lead
    - Lag
Outbound left leas: Inbound lags (pattern 1)

Inbound: $L_i \rightarrow G_i \rightarrow R_i$

Outbound: $\bar{G}_i \rightarrow \bar{L}_i \rightarrow R_i$

$r_i = R_i + L_i$
$\bar{r}_i = R_i + \bar{L}_i$
$\bar{r}_i + \bar{G}_i = 1$

$G_i(\bar{G}_i) =$ outbound (inbound) green time for through traffic at $S_i$
$L_i(\bar{L}_i) =$ time allocated for outbound (inbound) left turn traffic at $S_i$
$R_i =$ common red time in both directions to provide for cross street movement at $S_i$
Outbound left lags: Inbound leads (pattern 2)

\[ G_i \quad L_i \quad R_i \]

Inbound

Outbound

\[ \bar{L}_i \quad \bar{G}_i \quad R_i \]

\[ r_i = R_i + L_i \]
\[ \bar{r}_i = R_i + \bar{L}_i \]
\[ \bar{r}_i + \bar{G}_i = 1 \]
\[ r_i + G_i = 1 \]

\( G_i (\bar{G}_i) \) = outbound (inbound) green time for through traffic at \( S_i \)

\( L_i (\bar{L}_i) \) = time allocated for outbound (inbound) left turn traffic at \( S_i \)

\( R_i \) = common red time in both directions to provide for cross street movement at \( S_i \)
Outbound left leads: Inbound leads (pattern 3)

\[ r_i = R_i + L_i \]
\[ \bar{r}_i = R_i + \bar{L}_i \]
\[ \bar{r}_i + \bar{G}_i = 1 \]
\[ r_i + G_i = 1 \]

\( G_i \) (\( \bar{G}_i \)) = outbound (inbound) green time for through traffic at \( S_i \)
\( L_i \) (\( \bar{L}_i \)) = time allocated for outbound (inbound) left turn traffic at \( S_i \)
\( R_i \) = common red time in both directions to provide for cross street movement at \( S_i \)
Outbound left lags: Inbound lags (pattern 4)

\[ r_i = R_i + L_i \]
\[ \bar{r}_i = R_i + \bar{L}_i \]
\[ \bar{r}_i + \bar{G}_i = 1 \]
\[ r_i + G_i = 1 \]

\[ G_i (\bar{G}_i) \] = outbound (inbound) green time for through traffic at \( S_i \)

\[ L_i (\bar{L}_i) \] = time allocated for outbound (inbound) left turn traffic at \( S_i \)

\[ R_i \] = common red time in both directions to provide for cross street movement at \( S_i \)
$\Delta_1$: the time from the center of $r_1$ to the next center of $r_1$
\[ \Delta_i = 0.5 \times [(2\delta_i - 1) \times L_i - (2\bar{\delta}_i - 1) \times \bar{L}_i] \]

\[ \delta_i + \bar{\delta}_i = 1 \]

\[ \delta_i, \bar{\delta}_i : 0 - 1 \text{ variables} \]

<table>
<thead>
<tr>
<th>Pattern</th>
<th>( \Delta_i )</th>
<th>( \delta_i )</th>
<th>( \bar{\delta}_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(-0.5(L_i + \bar{L}_i))</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>(0.5(L_i + \bar{L}_i))</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>(-0.5(L_i - \bar{L}_i))</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>(0.5(L_i - \bar{L}_i))</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
MILP-2

Find \( b, \bar{b}, z, w_i, \overline{w_i}, t_i, \bar{t}_i, \delta_i, \overline{\delta_i}, m_i \) to

Max (\( b + k\bar{b} \)), subject to

\[
(1 - k)\bar{b} \geq (1 - k)b
\]

\[
1/C_2 \leq z \leq 1/C_1
\]

\[
w_i + b \leq 1 - r_i
\]

\[
\overline{w_i} + b \leq 1 - \overline{r_i}
\]
\[
\begin{align*}
\left( \overline{w_i} + w_i \right) - \left( \overline{w_{i+1}} + w_{i+1} \right) + (t_i + \overline{t_i}) \\
+ \delta_i L_i - \delta_i \overline{L_i} - \delta_{i+1} L_{i+1} + \overline{\delta_{i+1}} \overline{L_{i+1}} - m_i \\
= (r_{i+1} - r_i) + (\tau_i + \overline{\tau_i}) \\
\end{align*}
\]

\[
\begin{align*}
\frac{(d_i / f_i)}{z} \leq t_i \leq \frac{(d_i / e_i)}{z} \\
\frac{(d_i / h_i)}{z} \leq \frac{(d_i / d_{i+1})}{t_{i+1}} - t_i \leq \frac{(d_i / g_i)}{z} \\
\frac{(\overline{d_i} / \overline{f_i})}{z} \leq \overline{t_i} \leq \frac{(\overline{d_i} / \overline{e_i})}{z} \\
\frac{(\overline{d_i} / \overline{h_i})}{z} \leq \frac{(\overline{d_i} / \overline{d_{i+1}})}{\overline{t_{i+1}}} - \overline{t_i} \leq \frac{(\overline{d_i} / \overline{g_i})}{z} \\
b, \overline{b}, z, w_i, \overline{w_i}, t_i, \overline{t_i} \geq 0 \\
\delta_i, \overline{\delta_i} : 1 - 0 \text{ variable} \quad m_i \text{ integer}
\]
MILP-1

The basic, symmetric, uniform-width bandwidth maximization problem

MILP-2

Extends the basic problem to include asymmetric bandwidths in opposing directions, variable left-turn phase sequence, as well as decisions on cycle time length and link specific progression speeds

MILP-3

Presents the new multi-band, multi-weight approach, which also incorporates all previous decision capabilities
Define a different bandwidth for each directional road section of the arterial
Time from right side of red at S2 to centerline of outbound greenband

\[ w_2 + 0.5b_2 \leq 1 - r_2 \]
\[ w_2 - 0.5b_2 \geq 0 \]

Centerline of green band
\[ \begin{align*}
    w_2 + 0.5b_2 & \leq 1 - r_2 \\
    w_2 - 0.5b_2 & \geq 0
\end{align*} \] \Rightarrow \quad 0.5b_2 \leq w_2 \leq (1 - r_2) - 0.5b_2
Outbound

\[ w_3 + 0.5b_2 \leq 1 - r_3 \]
\[ w_3 - 0.5b_2 \geq 0 \]
\[ \begin{align*}
  w_3 + 0.5b_2 & \leq 1 - r_3 \\
  w_3 - 0.5b_2 & \geq 0
\end{align*} \] \\
\Rightarrow \quad \begin{align*}
  0.5b_2 & \leq w_3 \leq (1 - r_3) - 0.5b_2
\end{align*}
\[ 0.5b_2 \leq w_2 \leq (1 - r_2) - 0.5b_2 \]

For inbound

\[ 0.5b_i \leq w_i \leq (1 - r_i) - 0.5b_i \]

\[ 0.5\bar{b}_i \leq \bar{w}_i \leq (1 - \bar{r}_i) - 0.5\bar{b}_i \]

\[ 0.5b_2 \leq w_3 \leq (1 - r_3) - 0.5b_2 \]

For inbound

\[ 0.5b_i \leq w_{i+1} \leq (1 - r_{i+1}) - 0.5b_i \]

\[ 0.5\bar{b}_i \leq \bar{w}_{i+1} \leq (1 - \bar{r}_{i+1}) - 0.5\bar{b}_i \]
The ratio constraint from MILP-2 is changed to reflect the multi-band situation

Max \( (b + k\bar{b}) \), subject to

\[
\begin{align*}
\bar{b} & \geq kb \quad \text{if } k < 1 \quad \text{(outbound favored)} \\
\bar{b} & \leq kb \quad \text{if } k > 1 \quad \text{(inbound favored)} \\
\bar{b} & = b \quad \text{if } k = 1 \quad \text{balanced progression)
\end{align*}
\]

\[
(1 - k)\bar{b} \geq (1 - k)kb
\]

\[
(1 - k_i)\bar{b}_i \geq (1 - k_i)k_ib_i
\]
MILP-3

- Objective function

\[
\max B = \frac{1}{n-1} \sum_{i=1}^{n-1} (a_i b_i + \overline{a_i b_i})
\]

\[
a_i = \left( \frac{v_i}{s_i} \right)^p \quad \overline{a_i} = \left( \frac{\overline{v_i}}{\overline{s_i}} \right)^p
\]

*where*

\(v_i(\overline{v_i}) = \text{directional volume on section } i, \text{ outbound (inbound)}\)

\(s_i(\overline{s_i}) = \text{saturation flow on section } i, \text{ outbound (inbound)}\)

\(p = \text{exponential power; the values } p = 0,1,2,4 \text{ were used}\)
To obtain an objective function value that is consistent with those used previously, we normalized the weighting coefficients to obtain

\[
\sum_{i=1}^{n-1} a_i = n - 1
\]

\[
\sum_{i=1}^{n-1} \frac{a_i}{n-1} = n - 1
\]
MILP-3

Find \( b_i, \bar{b}_i, z, w_i, \bar{w}_i, t_i, \bar{t}_i, \delta_i, \bar{\delta}_i, m_i \) to

\[
\max B = \frac{1}{n-1} \sum_{i=1}^{n-1} \left( a_i b_i + \bar{a}_i \bar{b}_i \right)
\]

Subject to

\[
(1 - k_i) \bar{b}_i \geq (1 - k_i) k_i b_i
\]

\[
\frac{1}{C_2} \leq z \leq \frac{1}{C_1}
\]
\[ 0.5b_i \leq w_i \leq (1 - r_i) - 0.5b_i \]

\[ 0.5b_i \leq w_{i+1} \leq (1 - r_{i+1}) - 0.5b_i \]

\[ 0.5\bar{b}_i \leq \bar{w}_i \leq (1 - \bar{r}_i) - 0.5\bar{b}_i \]

\[ 0.5\bar{b}_i \leq \bar{w}_{i+1} \leq (1 - \bar{r}_{i+1}) - 0.5\bar{b}_i \]
\[
(w_i + w_{i+1}) - (\bar{w}_{i+1} + w_{i+1}) + (t_i + \bar{t}_i) \\
+ \delta_i L_i - \bar{\delta}_i \bar{L}_i - \delta_{i+1} L_{i+1} + \bar{\delta}_{i+1} \bar{L}_{i+1} - m_i \\
= (r_{i+1} - r_i) + (\tau_i + \bar{\tau}_i) \\
(d_i / f_i)z \leq t_i \leq (d_i / e_i)z \\
(d_i / h_i)z \leq (d_i / d_{i+1})t_{i+1} - t_i \leq (d_i / g_i)z \\
(\bar{d}_i / \bar{f}_i)z \leq \bar{t}_i \leq (\bar{d}_i / \bar{e}_i)z \\
(\bar{d}_i / \bar{h}_i)z \leq (\bar{d}_i / \bar{d}_{i+1})\bar{t}_{i+1} - \bar{t}_i \leq (\bar{d}_i / \bar{g}_i)z \\
b, \bar{b}, z, w_i, \bar{w}_i, t_i, \bar{t}_i \geq 0 \\
\delta_i, \bar{\delta}_i : 1 - 0 \text{ variable} \quad m_i \text{ integer}
Q & A