Robust optimization for signal plans under time-varying flows
Outline of this study

• Introduction
• Background and Methodology
• Deterministic Optimization
• Scenario-based Optimization
• Interval Optimization
• Numerical test and Sensitivity Analysis
• Conclusions
Introduction

• Many studies generated the signal optimization models based on deterministic traffic demands. Such as Webster (1958), Allsop (1972).

• The optimal setting of a pre-timed signal control is sensitive to the input demand pattern.

• The reliability of the signal plan optimized by “average demand” become a questionable issue.
Introduction

• Heydecker (1987) investigated the variability in traffic flows in his model, assuming the distribution of demand is obtained from data.

• Yin (2008) developed three models to deal with the day-to-day demand variations.
Introduction

• In this study, we explored the study on the time-varying traffic flows.

Most efficient way: actuate signal control system.
The traffic volumes of an intersection can be indicated by:

\[ \xi = V + \sigma \]

- The fluctuation of \( V \) is referred to the day-to-day demand variations.
- In this study, we are dealing with the uncertainty of \( \sigma \) and a robust model is developed.
Methodology

A deterministic optimization of the form is introduced:

\[
\begin{align*}
\min & \quad D(\Phi, \xi) \\
\text{s.t.} & \quad F(\Phi) \in K
\end{align*}
\]

To consider the uncertainty of input data, a “decision environment” is characterized by Ben-Tal (1998):

- A feasible solution to robust model must be satisfied by the constraints of the worst case in the uncertain set P.
- Problem with fixed input, d is called “nominal” instance of the uncertain problem.

Uncertain problem P:

\[
\min \{D(\Phi, \xi): F(\Phi) \in K\}_{\xi \in \Gamma}
\]

Certain problem P*:

\[
\min \{\sup_{\xi \in \Gamma} D(\Phi, \xi): F(\Phi) \in K \quad \forall \xi \in \Gamma\}
\]
Deterministic optimization model

• As a basis of the robust optimization model
• Could be treated as a “nominal instance” of the uncertain problem.
Deterministic optimization model

• Input
  ➢ Length of time interval (Period T was divided into a set of time intervals)
  ➢ Number of time intervals
  ➢ Time-dependent demand pattern

• Objective
  ➢ Minimize the average delay of the intersection during time period T

• Decision variable
  ➢ Cycle length and green time of each phase
Objective function

Highway Capacity Manual (HCM) 2000 provide an equation to estimate the delay per vehicle for each lane group:

\[
d(q_i, T_k) = \left( \frac{C}{2} \right) \left\{ \frac{(1 - g_{i} / C)^2}{1 - \left[ \min(1, X_{ik}) \right] g_{i} / C} \right\} PF + 900\Delta T \left[ (X_{ik} - 1) + \sqrt{(X_{ik} - 1)^2 + \left( \frac{4X_{ik}}{c_i\Delta T} \right)} \right] + \frac{1800Q_{ik} (1 + u_{ik}) t_{ik}}{c_i\Delta T}
\]

\[
\text{min } D = \frac{\sum_{k} \sum_{i} q_{i,k} * d(q_{i,k}, T_k)}{\sum_{k} \sum_{i} q_{i,k}}
\]

\[
\text{S.t } c_{\text{min}} \leq c \leq c_{\text{max}}
\]

\[
g_{\text{min}} \leq g_{i} \leq c
\]

\[
\sum_{i} g_{i} + \sum_{i} L_{i} = c
\]
Scenario-based optimization

• Applicable with sufficient survey data.

Assume that a set of demand scenarios is given:

\[ \Omega = (\xi_1, \xi_2, \ldots, \xi_i, \ldots, \xi_n) \]

\[ \xi_i = (\lambda_1, \lambda_2, \ldots, \lambda_k, \ldots, \lambda_T) \]

Thus, in order to solve the problem as below:

\[
\min \{ \sup_{\xi \in \Omega} D(\Phi, \xi): F(\Phi) \in K \quad \forall \xi \in \Omega \}
\]

A min-max model is introduced:

\[
\min_{\xi \in \Omega} \max \{ D(\Phi, \xi): F(\Phi) \in K \}
\]
Scenario-based optimization

• However, worst-case optimization may make the problem too conservative.

• An alternative way is to minimize the problem whose collective probability of occurrence exceed $\alpha$

We reorder the scenarios as an increasing list based on their delay according to a given signal timing design:

$$L = (L_1, L_2, \ldots, L_\varphi, \ldots, L_n | D(L_1) \leq D(L_2) \cdots \leq D(L_n))$$

And the probability of occurrence is marked as $P = (p_{L_1}, p_{L_2}, \ldots, p_{L_n})$.

Let $L_\varphi$ be the maximum delay exceed only with probability $\alpha$, so: $\sum_{i=1}^{\varphi} p_i \geq \alpha$. The objective function is becoming:

$$\min \{D(\Phi, L_\alpha) : F(\Phi) \in K \}$$

Subject to $\sum_{i=1}^{\varphi} p_{L_i} \geq \alpha$.
Scenario-based optimization

Advantage:

• Easy to formulate and convenient for application

Disadvantage:

• Require a large amount of field data
Interval optimization

- Moore (1979), Alefeld and Herzberger (1983) have done the pioneering work on the interval analysis

Applying the theory of interval analysis, the uncertainty set of traffic is defined as following intervals:

\[ \xi = (\lambda_I^1, \lambda_I^2, \ldots, \lambda_I^N) \]

where \( \lambda_I^q = [\lambda_q, \lambda_q^C] = [\lambda_q^C - \Delta \lambda_q, \lambda_q^C + \Delta \lambda_q] \)

The mid-point and uncertainty of the interval are defined as:

\[ \lambda_q^C = m(\lambda_I^q) = (\lambda_q + \lambda_q^C) / 2 \]

\[ \Delta \lambda_q = rad(\lambda_I^q) = (\lambda_q - \lambda_q^C) / 2 \]
The fluctuations of traffic volumes over the time period T are given by:

\[ \lambda_{qk} = \lambda_q^C + e_{qk} \Delta \lambda_q \]

Traffic flow of lane group q in time interval k

\[ e_{qk} \in [-1, 1] \]

Therefore, an interval optimization problem can be described as follow:

\[
\min \ { D(\Phi, \xi^I): F(\Phi) \in K } 
\]

The problem is becoming a optimization problem with an interval input.
Instead of considering a significant day-to-day variation of demands, the mean of demand over time period $T$ is controlled by:

$$\frac{1}{T} \sum_{k=1}^{T} \lambda_{qk} \leq \lambda_{q}^{C} (1 + \delta_{q}) \quad \text{for each } q = 1, 2, \cdots, n;$$

The disturbance parameter for the average demand can also be written as:

$$\frac{1}{T} \sum e_{qk} \leq \delta \frac{\lambda_{q}^{C}}{\Delta \lambda_{q}} = \alpha \quad \text{for each } q = 1, 2, \cdots, n;$$
To prevent of making the solution too conservative, we define an open ball in Cartesian space with p-norm to restrict the variation of traffic arrivals:

$$B(r) = \{ e_q \in R^n : \frac{1}{T} \sum_{k=1}^{T} |e_{qk}|^p < r^p \} \text{ for each } q = 1, 2, \cdots, n;$$

- If $r = 0$, the problem would become a deterministic problem respect to the mid-point demand pattern;
- If $r = 1$, the open ball lost the effect.
Solution algorithm

Applying the cutting-plane solution algorithm introduced by Bazaraa (1993), and the process is stated as:

Step 0: Using the average demand as input to obtain an initial optimal signal plan $\Phi_1$ and set the iteration counter $i = 1$.

Step 1: Given $\Phi_i$, solve the following sub problem to find out the worst-case in the Cartesian space:

$$\xi_k = \arg\max_{\xi^I} \{d(\Phi_i, \xi^I)\}$$

s.t.
$$\frac{1}{T} \sum_{k=1}^{T} e_{qk}^p < r^p \text{ for each } q = 1, 2, \ldots, n;$$
$$\frac{1}{T} \sum_{k=1}^{T} e_{qk} \leq \alpha \text{ for each } q = 1, 2, \ldots, n;$$

Step 2: Using $\xi_k$ generated in Step 1, solve the following optimization problem to find out the new optimal signal plan:

$$\Phi_{i+1} = \arg\min \{D(\Phi_i, \xi_k)\}$$

Step 3: If $|\Phi_{i+1} - \Phi_i| \leq \alpha$, then stop, otherwise, go back to Step 1 and $i = i+1$. 

For each searched node, solve a LP relaxation problem and obtained the bound.

However, because of the non-linear objective function, it would be very hard to solve the LP relaxation problem.
Modification of relaxation problem

Note: Fixed cycle length, delay decrease with the increase of green time.

New relaxation problem:
Delay (CL, g1, max(g2), max(g3), max(g4))
max(g2) = CL-g1-min(g3)-min(g4)-Lost time

Bound obtained
Modification of relaxation problem

- The modified problem is more “relax” than the LP problem.
- However, it is much easier to solve.

<table>
<thead>
<tr>
<th>Number of feasible solutions</th>
<th>Searched Node</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>1240942</td>
<td>51474</td>
<td>4.15%</td>
</tr>
</tbody>
</table>
Numerical example

In this section, we apply both scenario-based model and interval optimization model to an intersection as showing in the figure, each arm has 1 left-turn lane, 3 through lane and right-turn lane.

A specific phasing plan and phasing sequence is given:

\[ \Phi_1: (D1, D5); \quad \Phi_2: (D2, D6); \]
\[ \Phi_3: (D3, D7); \quad \Phi_4: (D4, D8). \]
## Input data

<table>
<thead>
<tr>
<th>Lane group</th>
<th>Under-saturated</th>
<th>Over-saturated</th>
<th>Saturation flow rate</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Average demand</td>
<td>Variation interval</td>
<td>Average demand</td>
</tr>
<tr>
<td>1</td>
<td>200</td>
<td>[100,300]</td>
<td>400</td>
</tr>
<tr>
<td>2</td>
<td>600</td>
<td>[400,800]</td>
<td>900</td>
</tr>
<tr>
<td>3</td>
<td>150</td>
<td>[50,250]</td>
<td>300</td>
</tr>
<tr>
<td>4</td>
<td>700</td>
<td>[500,900]</td>
<td>1200</td>
</tr>
<tr>
<td>5</td>
<td>200</td>
<td>[50,350]</td>
<td>300</td>
</tr>
<tr>
<td>6</td>
<td>600</td>
<td>[400,800]</td>
<td>1000</td>
</tr>
<tr>
<td>7</td>
<td>100</td>
<td>[50,150]</td>
<td>200</td>
</tr>
<tr>
<td>8</td>
<td>500</td>
<td>[300,700]</td>
<td>800</td>
</tr>
</tbody>
</table>

**Other input:**

*Lost time – 3s per phase and 12s in total.*

*The bound of cycle length is (50s,160s).*

*Analysis period is set to be 1 hours and segmented into 5 time intervals.*
100 Samples are randomly generated as the traffic scenarios. The confidential level is set to be 90% and 70%.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Phase 1 (s)</th>
<th>Phase 2 (s)</th>
<th>Phase 3 (s)</th>
<th>Phase 4 (s)</th>
<th>Cycle length (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Under-Saturated</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average</td>
<td>7</td>
<td>13</td>
<td>7</td>
<td>13</td>
<td>52</td>
</tr>
<tr>
<td>90th percentile</td>
<td>11</td>
<td>11</td>
<td>8</td>
<td>11</td>
<td>53</td>
</tr>
<tr>
<td>70th percentile</td>
<td>11</td>
<td>11</td>
<td>5</td>
<td>11</td>
<td>50</td>
</tr>
<tr>
<td><strong>Over-Saturated</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average</td>
<td>19</td>
<td>34</td>
<td>13</td>
<td>34</td>
<td>112</td>
</tr>
<tr>
<td>90th percentile</td>
<td>22</td>
<td>37</td>
<td>13</td>
<td>34</td>
<td>118</td>
</tr>
<tr>
<td>70th percentile</td>
<td>19</td>
<td>31</td>
<td>13</td>
<td>31</td>
<td>106</td>
</tr>
</tbody>
</table>
## Comparison of average delay via different signal plans

<table>
<thead>
<tr>
<th></th>
<th>Average-case</th>
<th>70th percentile worst-case</th>
<th>90th percentile worst-case</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Under-Saturated</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average</td>
<td>25s (0%)</td>
<td>34s (0%)</td>
<td>40s (0%)</td>
</tr>
<tr>
<td>70th percentile</td>
<td>26s (+4%)</td>
<td>26s (-24%)</td>
<td>34s (-15%)</td>
</tr>
<tr>
<td>90th percentile</td>
<td>28s (+12%)</td>
<td>29s (-15%)</td>
<td>25s (-38%)</td>
</tr>
<tr>
<td><strong>Over-Saturated</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average</td>
<td>82s (0%)</td>
<td>138s (0%)</td>
<td>149s (0%)</td>
</tr>
<tr>
<td>70th percentile</td>
<td>83s (+1.0%)</td>
<td>102s (-26.1%)</td>
<td>124 (-16.8%)</td>
</tr>
<tr>
<td>90th percentile</td>
<td>84s (+2.3%)</td>
<td>117s (-15.2%)</td>
<td>109s (-26.9%)</td>
</tr>
</tbody>
</table>
Interval optimization test

The norm of Cartesian space $P$ is set to be 2 and the open ball is becoming a Euclidean space.
The disturbance parameter is 0.2.
The radius of the open ball is 0.9 and 0.6.

<table>
<thead>
<tr>
<th></th>
<th>Phase 1 g(s)</th>
<th>Phase 2 g(s)</th>
<th>Phase 3 g(s)</th>
<th>Phase 4 g (s)</th>
<th>Cycle length(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Under-Saturated</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average</td>
<td>7</td>
<td>13</td>
<td>7</td>
<td>13</td>
<td>52</td>
</tr>
<tr>
<td>90th percentile</td>
<td>11</td>
<td>11</td>
<td>8</td>
<td>11</td>
<td>53</td>
</tr>
<tr>
<td>70th percentile</td>
<td>11</td>
<td>11</td>
<td>5</td>
<td>11</td>
<td>50</td>
</tr>
<tr>
<td>Over-Saturated</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average</td>
<td>19</td>
<td>28</td>
<td>13</td>
<td>31</td>
<td>103</td>
</tr>
<tr>
<td>90th percentile</td>
<td>32</td>
<td>47</td>
<td>14</td>
<td>38</td>
<td>143</td>
</tr>
<tr>
<td>70th percentile</td>
<td>32</td>
<td>38</td>
<td>14</td>
<td>35</td>
<td>131</td>
</tr>
</tbody>
</table>
Comparison of average delay via different signal plans

<table>
<thead>
<tr>
<th></th>
<th>Under-Saturated</th>
<th></th>
<th>Worst case in B_{0.6}</th>
<th>Worst case in B_{0.9}</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Average</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Overall</td>
<td>21.730 (0.000)</td>
<td>29.528 (0.000)</td>
<td>42.195 (0.000)</td>
<td></td>
</tr>
<tr>
<td>r =0.6</td>
<td>23.941 (2.211)</td>
<td>26.744 (-2.784)</td>
<td>31.456 (-10.739)</td>
<td></td>
</tr>
<tr>
<td>r =0.9</td>
<td>27.152 (5.422)</td>
<td>28.554 (-0.974)</td>
<td>30.442 (-11.753)</td>
<td></td>
</tr>
<tr>
<td><strong>Over-Saturated</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average</td>
<td>51.848 (0.000)</td>
<td>210.821 (0.000)</td>
<td>255.223 (0.000)</td>
<td></td>
</tr>
<tr>
<td>r =0.6</td>
<td>70.927 (19.079)</td>
<td>155.130 (-55.691)</td>
<td>195.378 (-59.845)</td>
<td></td>
</tr>
<tr>
<td>r =0.9</td>
<td>77.077 (25.229)</td>
<td>169.856 (-40.965)</td>
<td>184.849 (-70.374)</td>
<td></td>
</tr>
</tbody>
</table>