10 月 28 日課堂摘要

Definition Given a sequence $\{p_n\}$ in a metric space X, consider a sequence $\{n_k\}$ of positive integers, such that $n_1 < n_2 < n_3 < \cdots$. Then the sequence $\{p_{n_i}\}$ is called a ***subsequence*** of $\{p_n\}$. If $\{p_{n_i}\}$ converges, its limit is called a ***subsequential limit*** of $\{p_n\}$.

Note If $\{n_k\}$ is a sequence of positive integers such that $n_1 < n_2 < n_3 < \cdots$, then

$$n_i \geq i, \ \forall i \in N.$$

Examples Let $X = R$.

1. The set of all subsequential limits of $\{(-1)^n\}$ is $\{1, -1\}$.

2. Let $a_n = \begin{cases} (-1)^{\frac{n}{2}}, & \text{if } n \text{ is even.} \\ \dfrac{1}{n}, & \text{if } n \text{ is odd.} \end{cases}$, the set of all subsequential limits of $\{a_n\}$ is $\{1, -1, 0\}$.

Questions :

1. If $\{p_n\}$ converges to p, is it true that every subsequence of $\{p_n\}$ converges to p ?
2. Suppose that every convergent subsequence of $\{p_n\}$ converges to p, is it true that $\{p_n\}$ converges to p ?

Theorem

(a) *If $\{p_n\}$ is a sequence in a compact metric space X, then there exists a subsequence of $\{p_n\}$ that converges to a point of X.*

(b) *Every bounded sequence in R^k has a convergent subsequence.*

Recall

Let X be a metric space, $E \subseteq X$. If E is compact, then every infinite subset of E has a limit point in E.

Outline of the proof :

(a) Let E be the range of $\{p_n\}$.

(i) If E is finite, then $\exists\, p \in E$ such that there are infinitely many terms that take on the value p. Hence, $\exists\ \{n_k\}$ with $n_1 < n_2 < \cdots$ such that

$$p_{n_1} = p_{n_2} = ... = p$$

(ii) If E is infinite, then E has a limit point p in X. Choose n_1 such that $d(p, p_{n_1}) < 1$, then we can choose $n_2 > n_1$ such that $d(p, p_{n_2}) < \frac{1}{2}$ (why ?). Inductively, we get a subsequence $\{p_{n_k}\}$ such that

$$d(p, p_{n_k}) < \frac{1}{k},\ \ \forall\ k \in N.$$

(b) Let E be a bounded sequence in R^k, then $\exists\ k$-cell I such that $E \subseteq I$. Since I is compact, (b) follows from (a).

CAUCHY SEQUENCES

Definition A sequence $\{p_n\}$ in a metric space X is said to a ***Cauchy sequence*** if for every $\varepsilon > 0$ there is an integer N such that $d(p_n, p_m) < \varepsilon$ if $n \geq N$ and $m \geq N$.

Examples

1. Let $X = R$, $\left\{\frac{1}{n}\right\}$ is a Cauchy sequence that converges.
2. Let $X = (0,1)$, $\left\{\frac{1}{n}\right\}$ is a Cauchy sequence that does not converge.

Remarks

1. If $\{p_n\}$ is a Cauchy sequence in a metric X, then $\{p_n\}$ is bounded.
2. In any metric space X, every convergent sequence is a Cauchy sequence.

Definition Let E be a subset of a metric space X, and let

$$S = \{d(p,q) \mid p \in E,\ q \in E\}.$$

If $\sup S$ exists, then we say the ***diameter*** of E, denoted by $\operatorname{diam} E$, is $\sup S$.

Remarks

1. E is bounded iff $S = \{d(p,q) \mid p \in E,\ q \in E\}$ is bounded in R (why ?).

   Thus, if E is bounded, then $\sup S$ exists and $\text{diam}\, E > 0$.

2. E is unbounded iff $S = \{d(p,q) \mid p \in E,\ q \in E\}$ is unbounded in R.

   Thus, if E is unbounded, then $\sup S$ does not exists. In this case, we say $\text{diam}\, E = \infty$.

3. Let $\{p_n\}$ be a sequence in a metric space X and $E_n = \{p_i \mid i \geq n\}$. Then $\{p_n\}$ *is* a Cauchy sequence *if and only if*

$$\lim_{n\to\infty} \text{diam}\, E_n = 0.$$

Theorem

(a) *If* $\bar{E}$ *is the closure of a set* E *in a metric space* X*, then*

$$\text{diam}\, \bar{E} = \text{diam}\, E.$$

(b) *If* K_n *is a sequence of compact sets in* X *such that* $K_n \supset K_{n+1}$ $(n = 1, 2, 3, \ldots)$ *and if*

$$\lim_{n\to\infty} \text{diam}\, K_n = 0,$$

then $\bigcap_{n=1}^{\infty} K_n$ *consists of exactly one point.*

Theorem

(a) *If* X *is a compact metric space and if* $\{p_n\}$ *is a Cauchy sequence in* X*, then* $\{p_n\}$ *converges to some point of* X*.*

(b) *In* R^k*, every Cauchy sequence converges.*

Definition A metric space in which every Cauchy sequence converges is said to be ***complete***.

Examples

1. All compact metric spaces.
2. All Euclidean spaces are complete.
3. Every closed subset E of a complete metric space X is complete.