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MAXWELL'S EQUATIONS

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1. Introduction. We shall consider Maxwell's equations

$$(1) \cdots \operatorname{div} \mathbf{B} = 0 \qquad (2) \cdots \operatorname{div} \mathbf{D} = \sigma$$

$$(3) \cdots \operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad (4) \cdots \operatorname{curl} \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t}$$

in a "non-inductive" medium; i.e., $\mathbf{E} = \mathbf{D}$ is the electric field vector, $\mathbf{B} = \mathbf{H}$ is the magnetic field vector, σ is the charge density, and $\mathbf{j}$ is the current density vector.

These equations are usually taken as axioms in electromagnetic field theory. (1) says that there are no magnetic charges. (2) is Gauss' law, stating that one can compute the total charge inside a closed surface by integrating the normal component of $\mathbf{D}$ or $\mathbf{E}$ over the surface. (3) is Faraday's law; a changing magnetic field produces an electric field. Finally, (4) is Ampere's law curl $\mathbf{H} = \mathbf{j}$ modified by Maxwell's term $\partial \mathbf{D}/\partial t$, stating that currents and changing electric fields produce magnetic fields. Equations (1) and (2) are relatively simple and easily understood while (3) and (4) seem much more sophisticated. It is comforting to know then, that **in a certain sense, Faraday's law** (3) **is a consequence of** (1), **while the Ampere-Maxwell law** (4) **is a consequence of Gauss' law** (2). The precise statement will be found in Section 4. This apparently is a "folk-theorem" of physics; I first ran across the statement of it in an article of J. A. Wheeler ([3], p. 84). The precise statement involves only the simplest notions of special relativity and the proof of the statement is an extremely simple application of the formalism of exterior differential forms and could be written down in a few lines. I prefer to preface the proof with a very brief summary of special relativity and of how electromagnetism fits into special relativity, mainly because most (but not all) treatments of this subject motivate their constructions by means of Maxwell's equations; from our view point this would be circular and far less appealing than the approach via the Lorentz force.

2. The Minkowski Space of Special Relativity. Space-time is a 4-dimensional manifold M^4 that is topologically just R^4. A point in space-time is called an "event" and a curve of events is called a "world line". There are given admissible coordinate systems for M^4 corresponding to "inertial observers". In terms of such a coordinate system we may write $M^4 = R \times R^3$, and an event has coordinates (t, x, y, z) or briefly $(t, \mathbf{r})$; sometimes we shall write instead (x^0, x^1, x^2, x^3). Let (t', x', y', z') be coordinates set up by another inertial observer (physically, an observer moving uniformly with respect to the first observer). The basic assumption of special relativity is the following: if both observers look at the same world line, then in general

$$dt^2 \neq dt'^2 \text{ and } dx^2 + dy^2 + dz^2 \neq dx'^2 + dy'^2 + dz'^2,$$

but they will agree that (for simplicity we shall put the velocity of light $= 1$ in this article)

$$dt^2 - (dx^2 + dy^2 + dz^2) = dt'^2 - (dx'^2 + dy'^2 + dz'^2).$$

That is

$$d\tau^2 = dt^2 - d\mathbf{r} \cdot d\mathbf{r} = dt^2 - (dx^2 + dy^2 + dz^2)$$

defines an "invariant line element" in M^4.

The world line representing the history of a particle always has $d\tau^2 > 0$ (that is, the particle must travel at a speed less than the speed of light). For such a world line we can introduce τ as a new parameter (it is called "proper time"; the "chronometric hypothesis" states that τ is the time kept by an atomic clock moving with the particle) and we then have the unit tangent vector (the "velocity 4-vector")

$$V = \frac{dx}{d\tau}, \quad \text{i.e., } V^i = \frac{dx^i}{d\tau} \quad i = 0, 1, 2, 3.$$

Since

$$\frac{d\tau}{dt} = \left(1 - \frac{d\mathbf{r}}{dt} \cdot \frac{d\mathbf{r}}{dt}\right)^{\frac{1}{2}} = (1 - v^2)^{\frac{1}{2}},$$

where $\mathbf{v} = d\mathbf{r}/dt$ is the classical velocity vector, we may write the vector V as the 4 tuple

$$V = \left(\frac{dt}{d\tau}, \frac{d\mathbf{r}}{d\tau}\right) = \gamma(1, \mathbf{v}) \text{ with } \gamma = (1 - v^2)^{-\frac{1}{2}}.$$

Each particle has a "rest mass" m_0 (an invariant, independent of coordinates), and one then forms the momentum 4 vector

$$P = m_0 V = m_0 \gamma(1, \mathbf{v}) = (m, m\mathbf{v}),$$

where $m = m_0 \gamma$ is called the mass of the particle as viewed from the given coordinate system. As $v \to 1$ (speed of light) we have that $m = m_0 \gamma \to \infty$. Note that the spatial part $m\mathbf{v}$ of P is the classical momentum of a particle whose mass is m.

Finally, one has the notion of the Minkowski force 4-vector

$$f = \frac{dP}{d\tau} = \left(\frac{dm}{d\tau}, \frac{d}{d\tau}(m\mathbf{v})\right) = \left(\frac{dm}{d\tau}, \gamma\frac{d}{dt}(m\mathbf{v})\right) = (f^0, \gamma\mathbf{f}_c).$$

The term $(d/dt)(m\mathbf{v})$ is nothing but the classical force $\mathbf{f}_c$. Because $P = m_0 V$ is a vector of constant length m_0, we must have that $f = dP/d\tau$ is orthogonal to V in the Lorentz metric defined by $d\tau^2$. Now the scalar product of two 4 vectors $(A^0, \mathbf{A})$ and $(B^0, \mathbf{B})$ is given by

$$(A^0, \mathbf{A}) \cdot (B^0, \mathbf{B}) = A^0 B^0 - \mathbf{A} \cdot \mathbf{B}.$$

Since $f \cdot V = 0$ we conclude $\gamma f^0 - \gamma^2 \mathbf{f}_c \cdot \mathbf{v} = 0$, or $f^0 = \gamma \mathbf{f}_c \cdot \mathbf{v}$. (Incidentally, if we then equate f^0 with $dm/d\tau = \gamma\, dm/dt$ we get $dm = \mathbf{f}_c \cdot \mathbf{v}dt = \mathbf{f}_c \cdot d\mathbf{r}$, the element of work done by the force; this is Einstein's equivalence of mass and energy.)

We have written f as a contravariant vector $f^i = dP^i/d\tau$ since P is a tangent vector to a world line of a particle. One defines the covariant form of f, written $\mathfrak{f}$, by means of the usual process

$$\mathfrak{f}_i = \sum_{j=0}^{3} g_{ij} f^j, \text{ i.e., } \mathfrak{f} = (f^0, -\gamma\mathbf{f}_c) = \gamma(\mathbf{f}_c \cdot \mathbf{v}, -\mathbf{f}_c), \tag{5}$$

where (g_{ij}) is the metric tensor for $d\tau^2$, a diagonal matrix with $(1, -1, -1, -1)$ down the diagonal.

3. The electromagnetic field. The electromagnetic field vectors $\mathbf{E}$ and $\mathbf{B}$ in ordinary 3-space are measured via the Heaviside-Lorentz force formula; the force on a unit charge test particle is given by

$$\mathbf{f}_L = \mathbf{E} + \mathbf{v} \times \mathbf{B}, \tag{6}$$

where $\mathbf{v}$ is the velocity vector of the test charge. One determines $\mathbf{E}$ by measuring the force when the test particle is at rest, $\mathbf{v} = 0$. Having determined $\mathbf{E}$, one then measures the force when the test charge is given velocities in several directions, and so $\mathbf{B}$ is determined. *Thus the Heaviside-Lorentz force formula actually serves to define the fields* $\mathbf{E}$ *and* $\mathbf{B}$.

Einstein's study of special relativity apparently originated in essentially looking at equation (6) when one uses a coordinate system in 3-space that is moving uniformly with respect to the given system. For example, if one has $\mathbf{E} = 0$ and $\mathbf{B}$ is a constant field parallel to the z axis in a "fixed system", then when one passes to a system moving with speed v in the x direction, a unit test charge "at rest" in the moving system will suffer a force $\mathbf{v} \times \mathbf{B}$, and hence the moving observer will claim the existence of an electric field $\mathbf{E}' \neq 0$. Thus $\mathbf{E}$ and $\mathbf{B}$ individually may not be conceived as being vectors when one uses moving coordinate systems. To see how $\mathbf{E}$ and $\mathbf{B}$ should be joined together we can proceed as follows.

Suppose the unit test charge is moving along a curve in 3-space under the influence of a purely electromagnetic field. The classical force is then the Heaviside-Lorentz force, $\mathbf{f}_c = \mathbf{f}_L = \mathbf{E} + \mathbf{v} \times \mathbf{B}$. The covariant Minkowski 4 force is then, from equation (5)

$$\mathfrak{f} = (f^0, -\gamma\mathbf{f}_L) = \gamma(\mathbf{f}_L \cdot \mathbf{v}, -(\mathbf{E} + \mathbf{v} \times \mathbf{B}))$$
$$= \gamma(\mathbf{E} \cdot \mathbf{v}, -(\mathbf{E} + \mathbf{v} \times \mathbf{B})).$$

Writing out the components in full, one sees that there is a unique matrix $(\mathscr{F}_{ij})$ such that

$$\mathfrak{f}_i = -\sum_{j=0}^{3} \mathscr{F}_{ij} V^j,$$

namely

$$(\mathscr{F}_{ij}) = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix},$$

which is called the Minkowski electromagnetic field tensor. Since this matrix is skew-symmetric it defines an exterior form (apparently first considered by Hargreaves [see **4**, p. 250]).

$$\mathscr{F} = \sum_{i<j} \mathscr{F}_{ij} dx^i \wedge dx^j = \mathscr{E} \wedge dt + \mathscr{B}, \tag{7}$$

where $\mathscr{E} = E_x(t,\mathbf{r})\,dx + E_y(t,\mathbf{r})\,dy + E_z(t,\mathbf{r})\,dz = \mathbf{E} \cdot d\mathbf{r}$ and

$$\mathscr{B} = B_x(t,\mathbf{r})\,dy \wedge dz + B_y(t,\mathbf{r})\,dz \wedge dx + B_z(t,\mathbf{r})dx \wedge dy.$$

We shall refer to the book of Flanders [**2**] and the book of Y. Choquet-Bruhat [**1**] for any questions concerning differential forms and also for the uses of the form $\mathscr{F}$ in electromagnetic situations.

The 2-form $\mathscr{F} = \mathscr{E} \wedge dt + \mathscr{B}$ has been constructed using a particular inertial coordinate system (t, x, y, z). However, since all inertial coordinate systems will agree to use the Heaviside-Lorentz force formula to define the fields **E** and **B** in their coordinate systems we conclude that, in fact, *$\mathscr{F}$ is a well-defined exterior form on the Minkowski space M^4*. This has profound consequences. For instance, if (t', x', y', z') are coordinates used by another inertial coordinate system moving uniformly with respect to the first along the x axis, their coordinates are (as is well known) related by a Lorentz transformation

$$x' = \gamma(x - vt),\ y' = y,\ z' = z,\ t' = \gamma(t - vx). \tag{8}$$

Then, from the fact that $\mathscr{F}$ is well defined, we have $\mathscr{F} = \mathscr{E} \wedge dt + \mathscr{B} = \mathscr{E}' \wedge dt'$

$+ \mathscr{B}'$. Substituting equations (8) into $\mathscr{E}' \wedge dt' + \mathscr{B}'$ and comparing with $\mathscr{E} \wedge dt + \mathscr{B}$, one gets the well-known transformation laws for the fields **E** and **B**

$$\begin{aligned} E_x &= E'_x & B_x &= B'_x \\ E_y &= \gamma(E'_y + vB'_z) & B_y &= \gamma(B'_y - vE'_z) \\ E_z &= \gamma(E'_z - vB'_y) & B_z &= \gamma(B'_z + vE'_y). \end{aligned}$$

4. Maxwell's equations. Maxwell's source-free equations (1) and (3) are equivalent to the equation

$$d\mathscr{F} = 0,$$

where d is the exterior differential operator on Minkowski space M^4 (see [**1**] and [**2**] for details; we indicate briefly the essential ideas below). Symbolically, the exterior differential $\mathbf{d}$ for R^3 is

$$d = \mathbf{d} = dx \wedge \frac{\partial}{\partial x} + dy \wedge \frac{\partial}{\partial y} + dz \wedge \frac{\partial}{\partial z}$$

and the exterior differential d for M^4 is then $d = \mathbf{d} + dt \wedge (\partial/\partial t)$, all in terms of coordinates. Then

$$d\mathscr{F} = \left(\mathbf{d} + dt \wedge \frac{\partial}{\partial t}\right)(\mathscr{E} \wedge dt + \mathscr{B}) = \left(\mathbf{d}\mathscr{E} + \frac{\partial \mathscr{B}}{\partial t}\right) \wedge dt + \mathbf{d}\mathscr{B},$$

where by $\partial\mathscr{B}/\partial t$ we mean the 2-form $(\partial B_x/\partial t)\, dy \wedge dz + \cdots$. Thus

$$d\mathscr{F} = 0 \text{ iff } \begin{cases} \mathbf{d}\mathscr{B} = 0, \text{ i.e., } \operatorname{div} \mathbf{B} = 0 \\ \qquad\text{and} \\ \mathbf{d}\mathscr{E} + \dfrac{\partial \mathscr{B}}{\partial t} = 0, \text{ i.e., } \operatorname{curl} \mathbf{E} + \dfrac{\partial \mathbf{B}}{\partial t} = 0. \end{cases}$$

THEOREM. *Suppose that each inertial observer observes that* div $\mathbf{B} = 0$. *Then* $d\mathscr{F} = 0$ *and hence each observer also observes that* curl $\mathbf{E} = -(\partial \mathbf{B}/\partial t)$.

Proof. An inertial observer gives rise to a coordinate system (t, x, y, z) for Minkowski space, i.e., M^4 becomes the product $R \times R^3$. For fixed t_0 we have the inclusion map $\phi: R^3 \to R \times R^3$ sending R^3 onto the spatial section, $(x, y, z) \to (t_0, x, y, z)$. The observer puts $\mathscr{F} = \mathscr{E} \wedge dt + \mathscr{B}$. The restriction of $\mathscr{F}$ to his spatial section is simply $\phi^*\mathscr{F}$ (see [**2**], p. 23). But ϕ^* applied to forms merely says, put $t = t_0$ and $dt = 0$, and so

$$\phi^*\mathscr{F} = \phi^*\mathscr{B} = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy.$$

By hypothesis $d(\phi^*\mathscr{B}) = \mathbf{d}(\phi^*\mathscr{B}) = (\operatorname{div} \mathbf{B})\, dx \wedge dy \wedge dz = 0$, and so $d(\phi^*\mathscr{F}) = 0$ for each inertial observer. But d is a "natural" operation, i.e., $d(\phi^*\mathscr{F}) = \phi^*(d\mathscr{F})$. Thus the exterior 3-form $d\mathscr{F}$ has the property that its restriction to the spatial section

(t_0, x, y, z) is the zero-form, and this is true for all t_0 and for all inertial observers. We claim that this implies $d\mathcal{F} = 0$.

Being a 3-form in 4-space, $d\mathcal{F}$ has the form

$$d\mathcal{F} = N^0 dx \wedge dy \wedge dz - N^1 dt \wedge dy \wedge dz + N^2 dt \wedge dx \wedge dz - N^3 dt \wedge dx \wedge dy$$

for some coefficients N^0, N^1, N^2, N^3. In terms of the 4-vector N whose components are these coefficients, we can write $d\mathcal{F}$ as the "interior product" (see [**1**], p. 149)

$$d\mathcal{F} = i_N dt \wedge dx \wedge dy \wedge dz.$$

We claim that $N = 0$, hence $d\mathcal{F} = 0$. Suppose $N \neq 0$; then one can easily find three vectors e_1, e_2, and e_3 such that N, e_1, e_2, e_3 are linearly independent on M^4 and e_1, e_2, and e_3 are tangent to the spatial section of some inertial observer. (If N is not in the x, y, z plane, choose e_1, e_2, e_3 in this plane. If N is in the x, y, z plane we may assume it is along the x axis; then e_1, e_2, e_3 can be chosen along the x', y', z' axes of the coordinate system defined by the Lorentz transformation (8).) But $d\mathcal{F}$, as a multilinear function on vector triples, has value

$$d\mathcal{F}(e_1, e_2, e_3) = dt \wedge dx \wedge dy \wedge dz(N, e_1, e_2, e_3) \neq 0$$

since N, e_1, e_2, e_3 are linearly independent. This contradicts the fact that $d\mathcal{F} = 0$ on any spatial section of any inertial observer. This concludes the proof that $d\mathcal{F} = 0$.

THEOREM. *Suppose that each inertial observer observes that* div $\mathbf{E} = \sigma$. *Then each observer also observes that* curl $\mathbf{B} = \mathbf{j} + \partial\mathbf{E}/\partial t$.

Proof. Let $*$ denote the Hodge star operator taking p forms on M^4 into $(4 - p)$ forms (see [**1**] or [**2**]). Since $\mathcal{F} = \mathcal{E} \wedge dt + \mathcal{B}$ is a well defined 2-form on M^4, so is

$$\mathcal{G} = *\mathcal{F} = -\mathcal{H} \wedge dt + \mathcal{D},$$

where $\mathcal{H} = B_x dx + B_y dy + B_z dz = \mathbf{H} \cdot d\mathbf{r}$, and $\mathcal{D} = E_x dy \wedge dz + E_y dz \wedge dx + E_z dx \wedge dy$. (Here we are glossing over the question of orientations; however both the forms $\mathcal{G}$ and $\mathcal{S}$ defined below are in fact forms of "odd kind" in the sense of de Rham, and so no difficulties arise.)

There is also the notion of rest charge density σ_0, an invariant for all observers. The 4-vector $J = \sigma_0 V = (\sigma_0\gamma, \sigma_0\gamma\mathbf{v}) = (\sigma, \mathbf{j})$ is called the current 4-vector density, $\sigma = \sigma_0\gamma$ is the charge density, and $\mathbf{j} = \sigma\mathbf{v}$ is the classical current density vector. The exterior 3-form

$$\mathcal{S} = i_J dt \wedge dx \wedge dy \wedge dz = \sigma dx \wedge dy \wedge dz - (j_x dy \wedge dz + j_y dz \wedge dx + j_z dx \wedge dy) \wedge dt$$

is called the current 3-form. A simple calculation (see [**1**], [**2**]) as was done in our first Theorem shows that

$$d\mathcal{G} = \mathcal{S} \text{ iff } \begin{cases} \text{div } \mathbf{E} = \sigma \\ \quad\text{and} \\ \text{curl } \mathbf{B} = \mathbf{j} + (\partial\mathbf{E}/\partial t). \end{cases}$$

Again let $\phi: R^3 \to R \times R^3$ be defined by taking the spatial section of an inertial observer. Then

$$\phi^*(d\mathscr{G} - \mathscr{S}) = d\phi^*\mathscr{G} - \phi^*\mathscr{S} = d\phi^*\mathscr{D} - \phi^*\mathscr{S} = (\operatorname{div}\mathbf{E} - \sigma)dx \wedge dy \wedge dz$$

vanishes, by hypothesis, for each inertial observer. By the same reasoning as before one concludes that $d\mathscr{G} - \mathscr{S} = 0$, and we are finished.

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EVERYWHERE DIFFERENTIABLE, NOWHERE MONOTONE, FUNCTIONS

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The purpose of this paper is to construct an example of a real-valued function on R that is everywhere differentiable but is monotone on no interval and to examine further peculiar properties of any such function. Examples of such functions are seldom given, or even mentioned, in books on real analysis. The first explicit construction of such a function was given by Köpcke (1889). An example due to Pereno (1897) is reproduced in [**1**], pp. 412–421. We believe that the present construction, which grew out of a question asked of the first named author by the second, is shorter, more elementary, and easier to understand than any that we have seen. We proceed through a sequence of easy lemmas.

LEMMA 1. *Let r and s be real numbers.*
(i) *If $r > s > 0$, then $(r-s)/(r^2-s^2) < 2/r$.*
(ii) *If $r > 1$ and $s > 1$, then $(r+s-2)/(r^2+s^2-2) < 2/s$.*

Proof. Assertion (i) is obvious. Inequality (ii) is equivalent to

$$(r-s)^2 + (r-1)(s-1) + r^2 + r + 3s > 5.$$

But this too is obvious when $r > 1$ and $s > 1$.