Weierstraß*

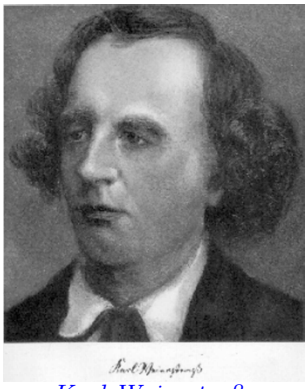
Volker Runde†

Each university and each department develops a peculiar kind of folklore—anecdotes about those of its graduates (or dropouts) that somehow managed to become famous (or notorious). Very often, there is an element of glee to these stories: "Well, he may now be a government minister, but *I* flunked him in calculus!" And also very often, it is impossible to tell the truth from the legend.

Karl Weierstraß

When I was a math student at Münster, Germany in the 1980s, such anecdotes centered mainly around two people: Gerd Faltings, the first and only German to win the Fields medal[1], mathematics' equivalent of the Nobel prize; and Karl Weierstraß, the man who (besides many other mathematical accomplishments) introduced ε and δ into calculus. Weierstraß had been a student at Münster in the 1830s and 1840s. There was no one around anymore who knew anybody who had known anybody who had known anybody who had known Weierstraß, but this didn't prevent the folklore from blooming. According to his legend, Weierstraß flunked out of law school because he spent most of his time there drinking beer and doing mathematics. Then he worked for more than ten years as a school teacher in remote parts of Prussia, teaching not only mathematics, but also subjects like botany, calligraphy, and physical education. Finally, when almost 40 years old, he became a famous mathematician, and was eventually appointed a professor at Berlin—without ever having received a PhD. This story may sound wild, and in some ways it simplifies the facts, but it is not far from the truth.

Karl Theodor Wilhelm Weierstraß was born on October 31, 1815, in the village of Ostenfelde, which is located in what was then the Prussian province of Westphalia. A street and an elementary school in Ostenfelde are named after him, and his birth house—still occupied today—is listed in a local tourist guide. His father, who worked for Prussia's customs and taxation authorities, was sent from one post to the next within short periods of time. For the first 14 years of Karl Weierstraß' life, his family was more or less constantly on the move. In 1829, Karl's father obtained an assistant's position at the tax office in the city of Paderborn (also in Westphalia), and the family could finally settle down. Young Karl enrolled at the local Catholic *Gymnasium* in Paderborn, where he excelled not only in mathematics, but also in German, Latin, and Greek. Not only was he a strong student, he was also quite capable of putting his brains to work on much more practical matters. At age 15, Karl contributed to his family's income by doing bookkeeping for a wealthy merchant's widow.



Karl Weierstraß as a young man

Throughout his life, Weierstraß Senior suffered from the knowledge that he did not have the right education to rise to a rank in the Prussian civil service that would have better suited his abilities. Instead, he had to content himself with relatively low-level positions, not very challenging and not very well paid. Like many a father in this situation, he was determined to prevent such a fate befalling his bright eldest son. When Karl graduated in 1834, his father decided to send him to Bonn to study *Kameralistik* (a combination of law, finance, and administration). Being a dutiful son, Karl went...

...and did all he could to sabotage the life his father had planned for him. He joined a *schlagende Verbindung*, a kind of student fraternity typical of German universities in the 19th century. Besides keeping the brewing industry busy, fraternity members engaged in a peculiar ritual: the *Mensur*, a swordfight with a peculiar twist. Unlike in today's athletic competitions, the students fought with sharp sabers. They wore protective gear that covered most of their bodies—except the cheeks. During a *Mensur*, the opponents tried to inflict gashes on one another's cheeks. The scars were borne with pride as signs of honour and manhood.[2] Almost two metres tall, quick on his feet, and with strong arms, Karl Weierstraß was a fearsome swordsman. His face remained unscarred, and after a while nobody was keen on challenging him anymore. Having escaped from under his father's tutelage, he spent his years at Bonn drinking beer and wielding the saber—and seriously studying mathematics. Although he was not enrolled in mathematics, he read some of the most advanced math books of his time. In 1838, when it was time for him to take his exams, he simply dropped out.

His family was desperate. They had made considerable financial sacrifices to secure a better future for Karl, who had let them down. Having wasted four years of his life, he needed a bread-winning degree, and fast. So, in 1839,

* The last letter is not a β, but an "ß", a letter unique to the German alphabet, which is pronounced like an "s". Books written in English usually spell the name "Weierstrass."

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[1] If you want to know more about the Fields medal: there is an article on it—*The Top Mathematics Award* by Florin Diacu—in the June 2001 issue of *π in the Sky*.

[2] If you find such ideas of honour and manhood absolutely revolting, you're absolutely right.

he enrolled at the *Akademie* in Münster, the forerunner to today's university, to become a secondary school teacher. Although this was not really a university, but rather a teacher training college, they had one good mathematician teaching there—Christoph Gudermann. He is said to have been an abysmal teacher: very often, he had just one student sitting in his class—Karl Weierstraß. In 1840, Weierstraß graduated. His thesis was so good that Gudermann believed it to be strong enough for a doctoral degree. However, the *Akademie* was not *really* a university; it did not have the right to grant doctorates. So, instead of receiving a doctorate and starting an academic career, Weierstraß left the *Akademie* as a mere school teacher.

His first job (probationary) was in Münster. One year later, he was sent to Deutsch Krona[3] in the province of West Prussia as an auxiliary teacher, then, in 1848, to Braunsberg[4] in East Prussia. Of course, he taught mathematics, but also physics, geography, history, German, and—believe it or not—calligraphy and physical education. Besides the demands of working full time as a teacher and having a social life (remember, he liked beer), he found time to do research in mathematics. During his time in Braunsberg, he published a few papers in his school's yearbook. High school year books are not exactly where people look for cutting edge research in mathematics, and so nobody noticed them. Then, in 1854, he published a paper entitled, "Zur Theorie der Abelschen Functionen" in a widely respected journal. I won't even make an attempt to explain what it was about. But unlike his previous work, this one *was* noticed.

It dawned on mathematicians all over Europe that the man who was probably the leading analyst of his day was rotting in a small East Prussian town, spending most of his time teaching youngsters calligraphy and physical education. On March 31, 1854, Weierstraß finally received a doctorate, an honorary one from the university of Königsberg.[5] In 1856, he accepted a position at the *Gewerbeinstitut* in Berlin, an engineering school, and a year later he joined the faculty of the University of Berlin as an adjunct professor. As a teacher, he attracted large audiences. Often, he taught in front of more than 200 students. In 1869, when he was almost 50 years old, Weierstraß was appointed full professor at the university of Berlin. In 1873 and 1874, he was *Rektor magnificus* of the university; in 1875, he became a knight of the order "Pour le Mérite" in the category of Arts and Sciences, the highest honour newly unified Germany could bestow upon one of its citizens; and, in 1885, on the occasion of his 70th birthday, a commemorative coin was issued in his honour.

The years of leading a double life as a secondary school teacher and a mathematical researcher took their toll on Weierstraß' health. A less vigorous man would probably have collapsed under the double burden much earlier. In 1850, Weierstraß began to suffer from attacks of dizziness, which culminated in a collapse in 1861. He had to pause for a year before he could teach again, and he never recovered fully. In 1890, at age 75, Weierstraß retired from teaching because of his failing health. The last years of his life were spent in a wheelchair. In 1897, he died.

Weierstraß published few papers—he was very critical toward his own work. But although he was a brilliant researcher, the greatest impact he had on mathematics was as a teacher. At Berlin, he repeatedly taught a two-year course on analysis, the predecessor of all modern introductions to calculus and analysis. Although he never wrote a textbook, notes taken in class by his students have survived and convey an impression of his lectures. Perhaps the longest lasting legacy of those lectures is their emphasis on rigour. When calculus was created in the 17th century, mathematicians did not worry about rigourously proving their results. For example, the first derivative dy/dx of a function $y = f(x)$ was thought of as a quotient of two "infinitesimals" (i.e., infinitely small quantities dy and dx). Nobody could really tell what infinitely small quantities were supposed to be, but mathematicians then didn't really care. The new mathematics enabled them to solve problems in physics and engineering that had been beyond the reach of the human mind before. So why bother with rigour? In the 18th century, mathematicians went so far as to proclaim that rigour was for philosophers and theologians, not for mathematicians. But with the lack of rigour, contradictory results cropped up with disturbing frequency—people often arrived at formulae that were obviously wrong. And, if a particular formula determines whether or not a bridge collapses, you don't want it to be wrong. Weierstraß realized that if calculus was to rest on solid foundations, its central notion, that of the limit, had to be made rigourous. He introduced the definition that (essentially) is still used today in classrooms:

> *A number y_0 is the limit of a function $f(x)$ as x tends to x_0 if, for each $\epsilon > 0$, there is $\delta > 0$ such that $|f(x) - y_0| < \epsilon$ for each x with $|x - x_0| < \delta$.*

Students may curse it, but it will not go away.

Weierstraß was not only an influential lecturer, but also one of the most prolific advisors of PhD theses of all time. There is a database on the Internet[6] that lists 31 PhD students of Weierstraß and 1,346 descendants (i.e., PhDs of PhDs of PhDs etc.) of Weierstraß. Interestingly, the two former students who generated the most folklore weren't his students in a technical sense.

Sofya Kovalevskaya was a young Russian noblewoman who had come to Germany to study mathematics. This alone was no small feat at a time when the very idea of a woman receiving a university education was revolutionary. For two years, she studied at Heidelberg, where authorities would not let her enroll officially, but eventually allowed her to attend lectures unofficially (provided the instructor did not object). Then she moved to Berlin to work with Weierstraß, only to find that she was not even allowed to audit lectures.

[3] Now Wałcz in Poland.
[4] Now Braniewo in Poland.
[5] Now Kaliningrad in Russia.

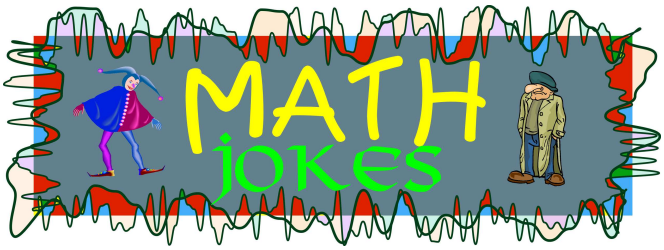
[6] The Mathematics Genealogy Project at http://hcoonce.math.mankato.msus.edu/ .

Sofya Kovalevskaya

This prompted Weierstraß, by all we know a politically conservative man, to tutor her privately. Since Kovalevskaya could not receive a doctorate from Berlin, Weierstraß used his influence to persuade the University of Göttingen to award her the degree in 1874. She spent the following nine years jobhunting. Being a woman didn't help. The best job she could find was teaching arithmetic at an elementary school. Finally, in 1883, she was offered a professorship at Stockholm, where she worked until her death in 1891 at the age of 41. Weierstraß and Kovalevskaya stayed in touch throughout her mathematical career. After her death, Weierstraß destroyed their correspondence. This fact, along with Kovalevskaya's striking beauty, gave rise to innuendos that she may have been more to Weierstraß (who never married) than just a student. Perhaps—but we don't know.

Karl Weierstraß in old age

Gösta Mittag-Leffler, another of the great mathematician's protegées, was also not Weierstraß' student strictly speaking. Already enrolled at the University of Uppsala, Sweden, he came to Berlin in 1875 to attend Weierstraß's lectures, which had an enormous impact on his mathematical development. He then returned to his native Sweden, where he received his doctorate. Over the years, Mittag-Leffler became indisputably the most influential mathematician of his time in Sweden. He made use of his clout to overcome the obstacles faced by Sofya Kovalevskaya regarding her appointment at Stockholm. What Mittag-Leffler is most famous for, however, is not a mathematical accomplishment, but a piece of mathematical folklore. To this day, mathematicians suffer quietly from the lack of a Nobel prize, and, some say, Mittag-Leffler is to blame—according to legend, the first version of Nobel's will mentioned a prize in mathematics. Then, Nobel found out that his wife had had an affair with Mittag-Leffler. Infuriated that his wife's lover could well be the first prize winner, Nobel changed his will and removed the math prize. That's a fine piece of juicy folklore, but nothing more; like Weierstraß, Nobel was a lifelong bachelor.



At the end of his course on mathematical methods in optimization, the professor sternly looked at his students and said, "There is one final piece of advice I'm going to give you now—whatever you have learned in my course, never, ever try to apply it to your personal lives!"

"Why?" the students asked.

"Well, some years ago, I observed my wife preparing breakfast, and I noticed that she wasted a lot of time walking back and forth in the kitchen. So, I went to work, optimized the whole procedure, and told my wife about it."

"And what happened?"

"Before I applied my expert knowledge, my wife needed about half an hour to prepare breakfast for the two of us. And now, it takes *me* less than fifteen minutes. . . ."

Q: What is an extroverted mathematician?

A: One who, in conversation, looks at the other person's shoes instead of at his own.

In a dark, narrow alley, a function and a differential operator meet, "Get out of my way or I'll differentiate you 'til you're zero!"

"Try it—I'm e^x. . . ."

Same alley, same function, but a different operator: "Get out of my way or I'll differentiate you 'til you're zero!"

"Try it—I'm e^x. . . ."

"Too bad. . . I'm d/dy."