NCTU - Human and Machine Lab

1

Chapter 3 – Kinematics and Inverse Kinematics

3.1 Introduction

NCTU - Human and Machine Lab

2

- **Definition:** The science of motion which treats motion without regard to the forces that cause it.
- **Trajectory:** Position, Velocity and Acceleration.
- **Link type:** Revolute, Prismatic, Spherical, Screw.

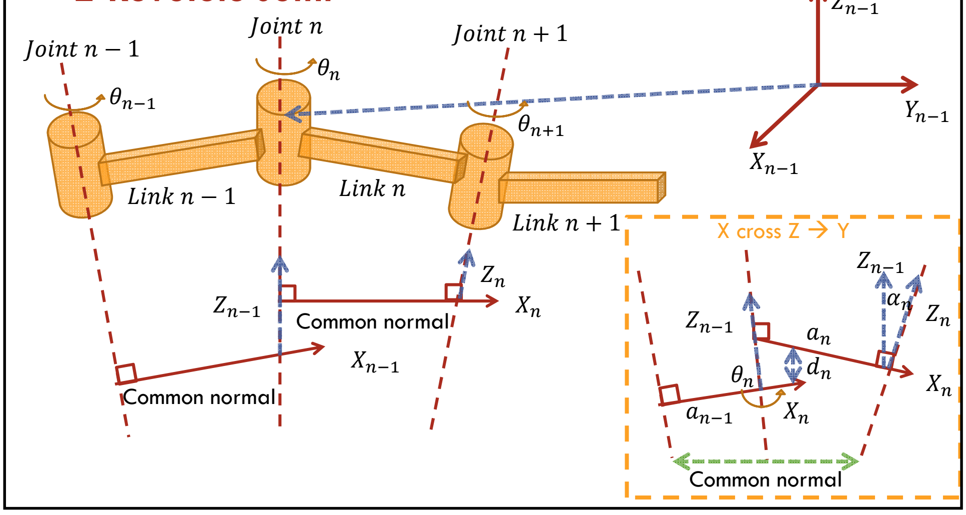
3.2 Link Connection

- $T_N = A_1 * A_2 * \cdots * A_N$ where N=number of links.

- Denavit-Hartenberg model (D-H model): (a, d, θ, α)
  - a_i: *link length*
  - d_i: *distance between two normals*
  - θ_i: *angle between two links*
  - α_i: *link twist*

3.2 Link Connection

- **Revolute Joint**



3.2 Link Connection

- Steps to establish the relationship between two coordinate systems
  - A. Select the coordinate frame
    - ① Define the origin of joint coordinates n at the intersection of the axis of the joint n+1 and the common normal between joint n and the joint n+1
    - ② The Z_n axis of joint coordinate n is aligned with the axis of joint n+1
    - ③ The X axis of joint coordinates n is the direction of the common normal between joint n and n+1

$$\vec{X}_n = \vec{Z}_{n-1} \times \vec{Z}_n \text{ or } \vec{X}_n = \vec{Z}_n \times \vec{Z}_{n-1}$$

3.2 Link Connection

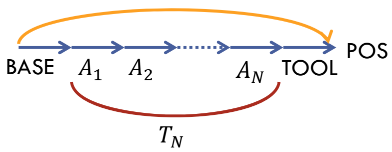
- Steps to establish the relationship between two coordinate systems
  - B. Find A_n transformation between coordinates frames n-1 and n
    - ① Rotate around Z_{n-1} and angle θ_n
    - ② Translate along Z_{n-1} a distance d_n
    - ③ Translate along the new X axis, X_n, a distance a_n
    - ④ Rotate around X_n a twist angle α_n

3.2 Link Connection

7 NCTU - Human and Machine Lab

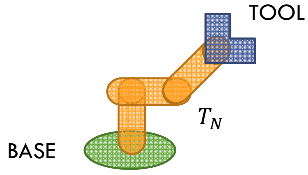
$$A_n = Rot(z,\theta_n)*Trans(0,0,d_n)*Trans(a_n,0,0)*Rot(x,\alpha_n)$$

$$= \begin{pmatrix} c\theta_n & -s\theta_n c\alpha_n & s\theta_n s\alpha_n & a_n c\theta_n \\ s\theta_n & c\theta_n c\alpha_n & -c\theta_n s\alpha_n & a_n s\theta_n \\ 0 & s\alpha_n & c\alpha_n & d_n \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



$$POS = BASE * T_N * TOOL$$

$$T_N = A_1 * A_2 * \cdots * A_N$$



3.2 Link Connection

8 NCTU - Human and Machine Lab

- In the case of prismatic joint, the origin of the coordinate frame for a prismatic joint is coincident with the next defined link origin.
- The zero position is defined when $d_n = 0, a_n = 0$
- A_n matrix is reduced to

$$A_n = \begin{pmatrix} c\theta_n & -s\theta_n c\alpha_n & s\theta_n s\alpha_n & 0 \\ s\theta_n & c\theta_n c\alpha_n & -c\theta_n s\alpha_n & 0 \\ 0 & s\alpha_n & c\alpha_n & d_n \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Where θ_n is a fixed number

3.2 Link Connection

- Remarks:
  - For revolute joint, θ is joint variable.
  - For prismatic joint, d is a joint variable.
  - In the case of prismatic joint, the length a_n has no meaning and is set to zero.
  - The origin of the coordinate frame for a prismatic joint is coincident with the next defined link origin. The zero position is defined when $d_n = 0$.

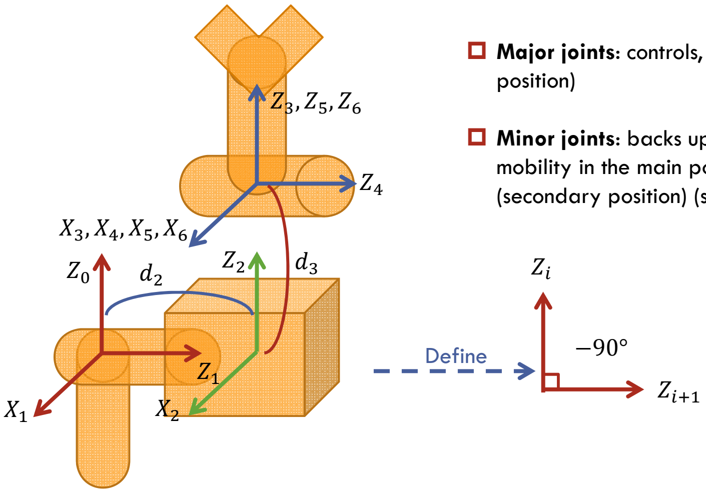
3.3 Example for Kinematics

A. Stanford Arm

Kinematic Table

Joint	θ	d	a	α	Joint variable
1	θ_1	0	0	$-90°$	θ_1
2	θ_2	d_2	0	$90°$	θ_2
3	$0°$	d_3	0	$0°$	d_3
4	θ_4	0	0	$-90°$	θ_4
5	θ_5	0	0	$90°$	θ_5
6	θ_6	0	0	$0°$	θ_6

3.3 Example for Kinematics



- **Major joints:** controls, position (main position)
- **Minor joints:** backs up the lack of mobility in the main position (secondary position) (spherical)

3.3 Example for Kinematics

$$A_1 = \begin{pmatrix} C_1 & 0 & -S_1 & 0 \\ S_1 & 0 & C_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} C_2 & 0 & S_2 & 0 \\ S_2 & 0 & -C_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_4 = \begin{pmatrix} C_4 & 0 & -S_4 & 0 \\ S_4 & 0 & C_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_5 = \begin{pmatrix} C_5 & 0 & S_5 & 0 \\ S_5 & 0 & -C_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_6 = \begin{pmatrix} C_6 & -S_6 & 0 & 0 \\ S_6 & C_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3.3 Example for Kinematics

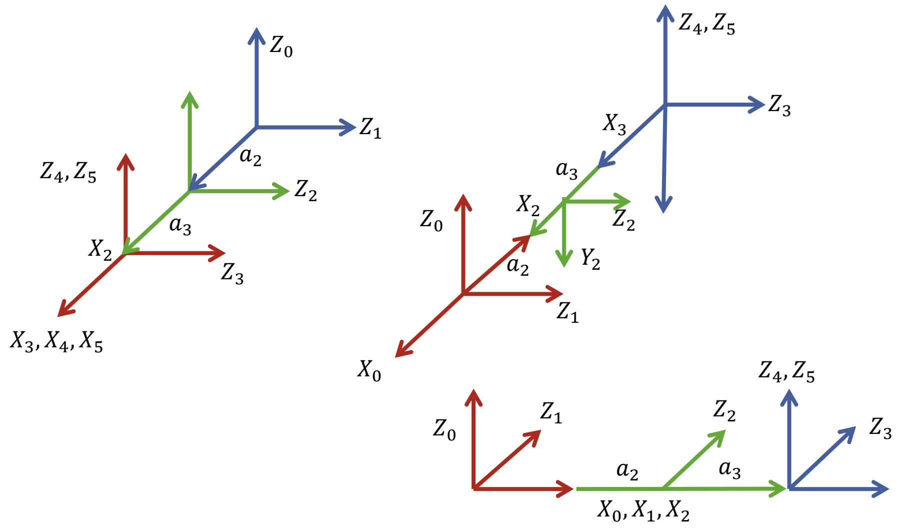
13 NCTU - Human and Machine Lab

B. Yasukawa Arm

Kinematic Table				
joint	θ	d	a	α
1	θ_1	0	0	$-90°$
2	θ_2	0	a_2	$0°$
3	θ_3	0	a_3	$0°$
4	θ_4	0	0	$90°$
5	θ_5	0	0	$0°$

3.3 Example for Kinematics

14 NCTU - Human and Machine Lab



3.3 Example for Kinematics

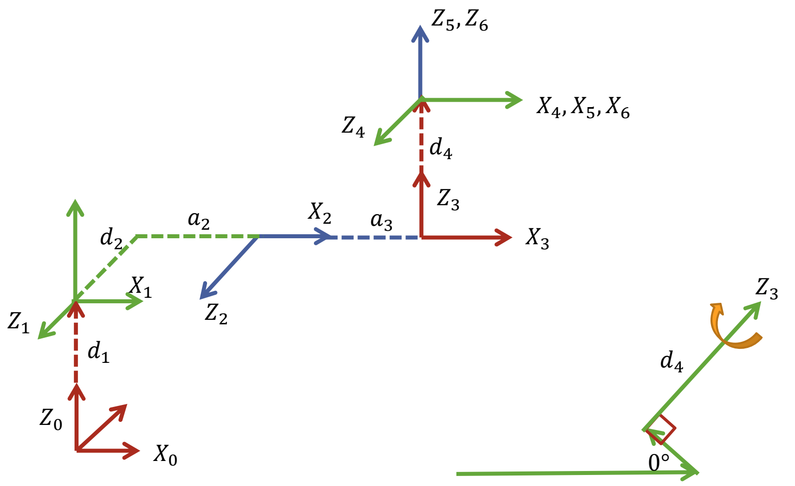
15 NCTU - Human and Machine Lab

c. PUMA 560

Kinematic Table				
Joint	θ	d	a	α
1	θ_1	d_1	0	$90°$
2	θ_2	d_2	a_2	$0°$
3	θ_3	0	a_3	$-90°$
4	θ_4	d_4	0	$90°$
5	θ_5	0	0	$-90°$
6	θ_6	0	0	$0°$

3.3 Example for Kinematics

16 NCTU - Human and Machine Lab



3.3 Example for Kinematics

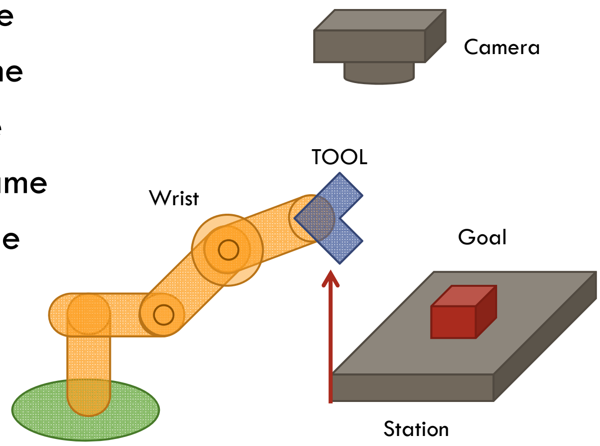
$$A_1 = \begin{pmatrix} C_1 & 0 & S_1 & 0 \\ S_1 & 0 & -C_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad A_2 = \begin{pmatrix} C_2 & -S_2 & 0 & a_2C_2 \\ S_2 & C_2 & 0 & a_2S_2 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} C_3 & 0 & -S_3 & a_3C_3 \\ S_3 & 0 & C_3 & a_3S_3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad A_4 = \begin{pmatrix} C_4 & 0 & S_4 & 0 \\ S_4 & 0 & -C_4 & 0 \\ 0 & 1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_5 = \begin{pmatrix} C_5 & 0 & -S_5 & 0 \\ S_5 & 0 & C_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad A_6 = \begin{pmatrix} C_6 & -S_6 & 0 & 0 \\ S_6 & C_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3.4 Frames of Convention

- Base frame
- Wrist frame
- Tool frame
- Station frame
- Goal frame



3.5 Inverse Kinematics

19 NCTU - Human and Machine Lab

- Existence:
  - Problem: given T_N solve θ_1 to θ_N
  - Basically, this is a nonlinear problem.
  - It can be solved by numerical methods (for N=6)
  - Do not have the closed-form solution for general types of robot manipulators
  - Then why industrial robots are designed to be wrist partitioned?

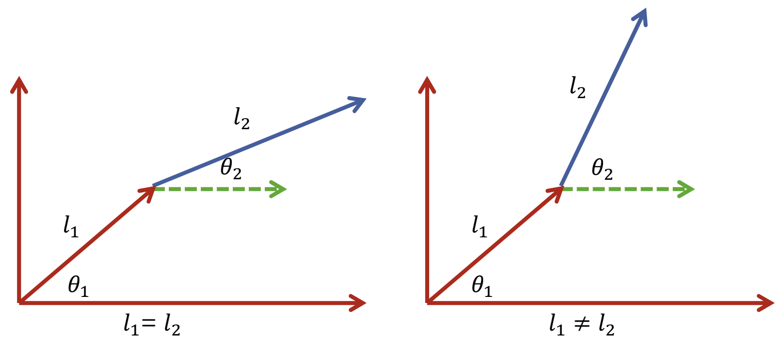
3.5 Inverse Kinematics

20 NCTU - Human and Machine Lab

- Workspace:
  - The volume of space which the end-effector of the manipulator can reach
- Dexterous Workspace:
  - The volume of space which the robot end-effector of the manipulator can reach with all orientations
- Reachable Workspace:
  - The volume of space which the robot end-effector of the manipulator can reach in at least one direction

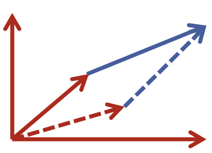
3.5 Inverse Kinematics

- For instance

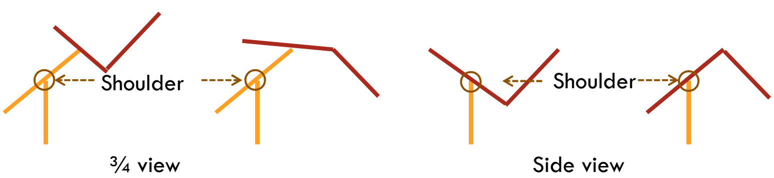


3.5 Inverse Kinematics

- Configuration
  - A position may be reached with several different configurations



- PUMA Case:
  - Four ways to reach the same wrist position. Another two configurations are present in the minor joints.

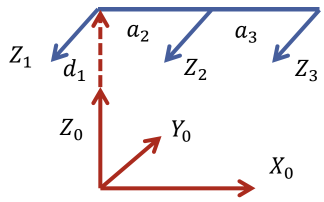


3.5 Inverse Kinematics

- Algebraic vs Geometric
  - For a 3R robot manipulator with the following space:

joint	d	a	α
1	d_1	0	90°
2	0	a_2	0°
3	0	a_3	0°

  - Find the T_3 matrix and solve its corresponding solutions by both the algebraic and geometric approaches



3.5 Inverse Kinematics

$$A_1 = \begin{pmatrix} C_1 & 0 & S_1 & 0 \\ S_1 & 0 & -C_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad A_2 = \begin{pmatrix} C_2 & -S_2 & 0 & a_2C_2 \\ S_2 & C_2 & 0 & a_2S_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad A_3 = \begin{pmatrix} C_3 & -S_3 & 0 & a_3C_3 \\ S_3 & C_3 & 0 & a_3S_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Geometric

$$T_3 = \begin{pmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Direction Position

$$= \begin{pmatrix} C_1C_2C_3 - C_1S_2S_3 & -C_1C_2S_3 - C_1S_2C_3 & S_1 & a_2C_1C_2 + a_3(C_1C_2C_3 - C_1S_2S_3) \\ S_1C_2C_3 - S_1S_2S_3 & -S_1C_2S_3 - S_1S_2C_3 & -C_1 & a_2S_1C_2 + a_3(S_1C_2C_3 - S_1S_2S_3) \\ S_2C_3 + C_2S_3 & -S_2S_3 + C_2C_3 & 0 & d_1 + a_2S_2 + a_3(S_2C_3 + C_2S_3) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3.5 Inverse Kinematics

A. Algebraic solution: From the fourth column (why not from the third column?)

$$\frac{P_y}{P_x} = \frac{S_1\left[a_2c_2 + a_3c(\theta_2+\theta_3)\right]}{C_1\left[a_2c_2 + a_3c(\theta_2+\theta_3)\right]}$$

- Check if $a_2c_2 + a_3c_{23} = 0$ if it is not 0 then solve θ_1, this will yield 2 solutions

$$T_3 = T_1 \cdot {}^1T_3 = A_1\, {}^1T_3$$

$$A_1^{-1}T_3 = {}^1T_3 = A_2 \cdot A_3$$

$$\begin{pmatrix} C_1 & S_1 & 0 & 0 \\ 0 & 0 & 1 & -d_1 \\ S_1 & -C_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} C_{23} & -S_{23} & 0 & a_2C_2 + a_3C_{23} \\ S_{23} & C_{23} & 0 & a_2S_2 + a_3S_{23} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3.5 Inverse Kinematics

NCTU - Human and Machine Lab

From:

$$p_xC_1 + p_yS_1 = a_2C_2 + a_3C_{23} \quad (1)$$

$$p_z - d_1 = a_2S_2 + a_3S_{23} \quad (2)$$

Let

$$f_1 = p_xC_1 + p_yS_1$$

$$f_2 = p_z - d_1$$

Rearrange (1) and (2)

$$f_1 - a_2C_2 = a_3C_{23} \quad (3)$$

$$f_2 - a_2S_2 = a_3S_{23} \quad (4)$$

Square and Sum (3) and (4)

$$f_1^2 + f_2^2 + a_2^2 - 2a_2f_1C_2 - 2a_2f_2S_2 = a_3^2$$

$$2a_2f_1C_2 + 2a_2f_2S_2 = f_1^2 + f_2^2 + a_2^2 - a_3^2$$

θ_2 can be found (2 solutions)

Square and Sum (1) and (2)

$$f_1^2 + f_2^2 = a_2^2 + a_3^2 + 2a_2a_3(C_2C_{23} + S_2S_{23})$$

$$= a_2^2 + a_3^2 + 2a_2a_3C_3$$

$$C_3 = \frac{f_1^2 + f_2^2 - a_2^2 - a_3^2}{2a_2a_3}$$

Derive the corresponding S_3, then find $tan\theta_3$, it should give 4 solution for 4 configurations. Then check the joint limits for each configuration.

3.5 Inverse Kinematics

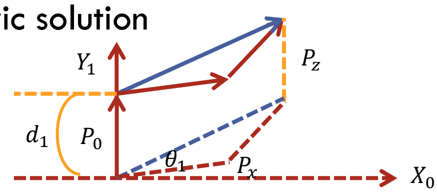
27 NCTU - Human and Machine Lab

B. **Geometric solution**

θ_1

$\tan\theta_1 = \frac{p_y}{p_x}$

(2 solutions)



$\theta_2 and \theta_3$

Let $R = \sqrt{{}^1p_x^2 + {}^1p_y^2}$

Rotate around Z_0 an angle θ_1 to remove the effect of $j^{\#}1$ rotation

$${}^1p_x = p_x C_1 + p_y S_1 = f_1$$

$${}^1p_y = p_z - d_1 = f_2$$

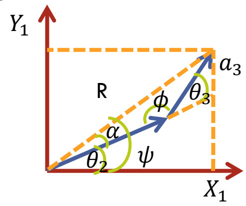
$$\theta_3 = \pi - \phi$$

$$\cos\phi = \frac{a_2^2 + a_3^2 - R^2}{2a_2 a_3}$$

$$\tan\psi = \frac{{}^1p_y}{{}^1p_x}$$

$$\cos\alpha = \frac{R^2 + a_2^2 - a_3^2}{2Ra_2}$$

$$\theta_2 = \psi - \alpha$$



3.6 PUMA 560

28 NCTU - Human and Machine Lab

- PUMA 560
  - Wrist partitioned type
    - Primary joints: for the positional workspace
    - Minor joint: for the orientation workspace

- The industrial robot manipulators usually consist of minor joint intersecting at the same point.

3.6 PUMA 560

- PUMA 560

Find A_1 to A_6 and T_6

$$T_6 = \begin{pmatrix} f_{11} & f_{12} & f_{13} & p_x \\ f_{21} & f_{22} & f_{23} & p_y \\ f_{31} & f_{32} & f_{33} & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix} = [\sim]$$

Kinematics Table

Joint	θ	d	a	α
1	θ_1	0	0	$-90°$
2	θ_2	0	a_2	$0°$
3	θ_3	d_3	a_3	$90°$
4	θ_4	d_4	0	$-90°$
5	θ_5	0	0	$90°$
6	θ_6	0	0	$0°$

Algebraic solution

$$A_1^{-1} \cdot T_6 = {}^1T_6 = A_2 \cdot A_3 \cdot A_4 \cdot A_5 \cdot A_6$$

$$\begin{pmatrix} C_1 & S_1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -S_1 & C_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_{11} & f_{12} & f_{13} & p_x \\ f_{21} & f_{22} & f_{23} & p_y \\ f_{31} & f_{32} & f_{33} & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix} = {}^1T_6 = \begin{pmatrix} \vdots & \vdots & \vdots & S_{23}d_4 + C_{23}a_3 + a_2C_2 \\ \vdots & \vdots & \vdots & -C_{23}d_4 + S_{23}a_3 + a_2S_2 \\ \vdots & \vdots & \vdots & d_3 \\ \vdots & \vdots & \vdots & 1 \end{pmatrix}$$

3.6 PUMA 560

$$-S_1 p_x + C_1 p_y = d_3$$

Let $p_x = p\cos\phi$

$p_y = p\sin\phi$

With $p = \sqrt{p_x^2 + p_y^2}, \ \phi = A\tan 2(p_y, p_x)$

$$C_1 S\phi - S_1 C\phi = \frac{d_3}{p}$$

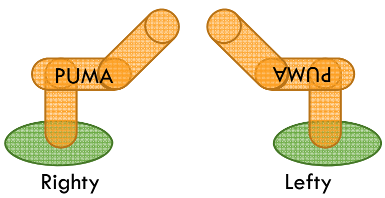
$$\sin(\phi - \theta_1) = \frac{d_3}{p}$$

$$\therefore \cos(\phi - \theta_1) = \pm\sqrt{1 - \frac{d_3^2}{p^2}}$$

$$\therefore \phi - \theta_1 = A\tan 2\left[\frac{d_3}{p}, \pm\sqrt{1 - \frac{d_3^2}{p^2}}\right]$$

$$\theta_1 = A\tan 2(p_y, p_x) - A\tan 2(d_3, \pm\sqrt{p_x^2 + p_y^2 - d_3^2})$$

- Two configurations:
  - (-) Righty: θ_2 moves the wrist in the positive Z_0 direction
  - (+) Lefty: θ_2 moves the wrist in the negative Z_0 direction



3.6 PUMA 560

31 NCTU - Human and Machine lab

From

$$C_1P_X + S_1P_Y = S_{23}d_4 + C_{23}a_3 + a_2C_2 \quad (1)$$
$$-P_Z = -C_{23}d_4 + S_{23}a_3 + a_2S_2 \quad (2)$$
$$-S_1P_X + C_1P_Y = d_3 \quad (3)$$

Square and Sum (1), (2) and (3)

$$a_3C_3 + d_4S_3 = \frac{P_X^2 + P_Y^2 + P_Z^2 - a_2^2 - a_3^2 - d_3^2 - d_4^2}{2a_2} = M$$
$$\therefore \theta_3 = A\tan 2(M, \pm\sqrt{a_3^2 + d_4^2 - M^2}) - A\tan 2(a_3, d_4)$$

- Two configurations:
  - $(-)$ {Righty and above / Lefty and below}
  - $(+)$ {Righty and below / Lefty and *above*}

3.6 PUMA 560

32 NCTU - Human and Machine lab

- Definiton:
  - Above (elbow wrist): position of the wrist of the righty (lefty) arm with respect the shoulder system has negative (positive) coordinates values along the Y_2 axis
  - Above (below wrist): position of the wrist of the righty (lefty) arm with respect the shoulder system has positive (negative) coordinates values along the Y_2 axis

3.6 PUMA 560

Get

$T_3^{-1} \cdot T_6 = {}^3T_6 = A_4 \cdot A_5 \cdot A_6$

$$\begin{pmatrix} C_1C_{23} & S_1C_{23} & -S_{23} & -a_2 - a_2C_3 \\ -S_1 & C_1 & 0 & -d_3 \\ C_1S_{23} & S_1S_{23} & C_{23} & -a_2S_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_{11} & f_{12} & f_{13} & p_x \\ f_{21} & f_{22} & f_{23} & p_y \\ f_{31} & f_{32} & f_{33} & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} - & - & - & 0 \\ - & - & - & 0 \\ - & - & - & d_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$C_1C_{23}P_x + P_yS_1C_{23} - P_zS_{23} = a_2 + a_2C_3 \quad \Rightarrow \theta_{23} \text{ can be solved}$$

$$C_1S_{23}P_x + P_yS_1S_{23} + P_zC_{23} = d_4 + a_2S_3 \quad \Rightarrow \theta_2 = \theta_{23} - \theta_3$$

Solve for $j^{\#}$4,5,6

From

$${}^3T_6 = \begin{pmatrix} C_4C_5C_6 - S_4S_6 & -C_4C_5S_6 - S_4C_6 & C_4S_5 & 0 \\ S_4C_5C_6 + C_4S_6 & -S_4C_5S_6 + C_4C_6 & S_4S_5 & 0 \\ -S_5C_6 & S_5S_6 & C_5 & d_4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (*)$$

3.6 PUMA 560

From C_4S_5, S_4S_5 in (*)

$$\tan\theta_4 = \frac{S_4S_5}{C_4S_5} = \frac{S_4}{C_4} \quad \text{If only } S_5 \neq 0$$

2 solutions depend on the signs of S_5

- Flip and non-Flip
- Wrist up and Wrist down

From ${}^4T_6 = A_5 \cdot A_6$

$${}^4T_6 = \begin{pmatrix} \cdots & \cdots & S_5 & 0 \\ \cdots & \cdots & -C_5 & 0 \\ \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & 0 & 1 \end{pmatrix}$$

θ_5 can be solved

From S_5, C_6, S_5, S_6 in (*), θ_6 can also be solved

When $\theta_5 = 0$, from (*)

We have $-C_4S_6 - S_4C_6 = -S_{46}$

$-S_4S_6 + C_4C_6 = C_{46}$

Then θ_{46} can be solved

Choose the previous θ_4

Then $\theta_6 = \theta_{46} - \theta_4$

Check joint ranges

3.6 PUMA 560

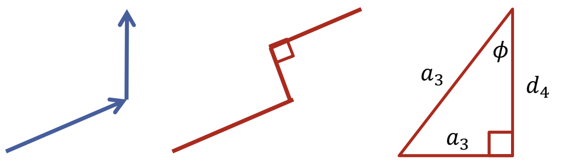
35 NCTU - Human and Machine Lab

- Geometric solutions:
  1. Choose the wrist point as the reference.
  2. Divide it into the position and orientation workspaces.

3.6 PUMA 560

36 NCTU - Human and Machine Lab

- In the positional work space, find P_ω from $POS = [n\ o\ a\ p]$
  a) Project the position into the (X_0, Y_0) plane and determine θ_1, also 2 configurations exist
  b) Rotate around Z_0 a $-\theta_1$ to remove the effect
  c) Angles $\theta_2\ and\ \theta_3$ can be determined similar to the case of

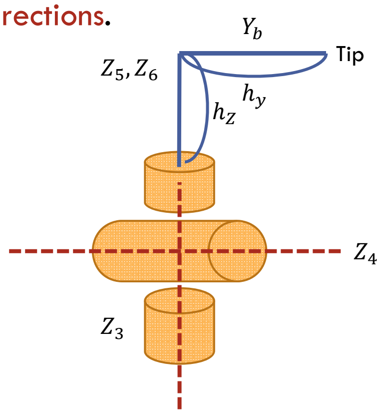


$\phi = \tan^{-1}\frac{a_3}{d_4}$

except that an offset a_3 exist. It can be incorporated into link3. Then an expanded link3 is formed

3.6 PUMA 560

- In the orientation workspace
  - The end-effector needs to have lengths in at least two different directions.



3.6 PUMA 560

- Assume: The end-effector h_y in the o_e direction and h_z in the a_e direction.
- First derive h_y and h_z represented in the wrist coordinates.
- Find ${}^w h_y$ and ${}^w h_z$
- ${}^w h_z$ can be used to determine the feasibilities of θ_4, θ_5. Then the effect of ${}^w h_z$ can be removed. ${}^w h_y$ can be used to determine the feasibility of θ_6.
- Practice: find the joint solutions for Stanford Arm

3.7 Repeatability and Accuracy

39 NCTU - Human and Machine Lab

- **Repeatability**: the precision with which a manipulator can return to a previous taught point.
- **Accuracy**: the precision with which a manipulator can reach a specified Cartesian point.
  - Related to the inverse kinematics and bounded by the repeatability
- Robot calibration (based on a small-error model)
  - A. Modeling: what causes the errors
    - Geometrical error
    - Non-geometrical
  - B. Measurement
  - C. Calibration model
  - D. Compensation
- Reference: papers related to Robot Calibration and Compensation