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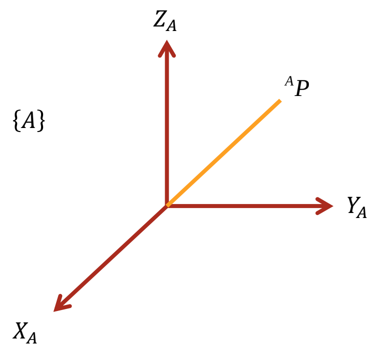
Chapter 2 – Spatial Representation and Transformation

2.1 Position, Orientation and Frames

- Position Vector

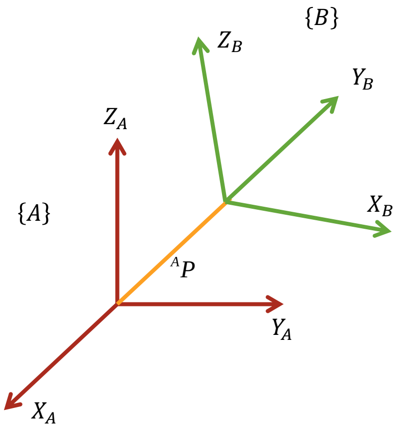
$$ {}^{A}P = \begin{bmatrix} P_X \\ P_Y \\ P_Z \\ \omega \end{bmatrix} $$

Usually $\omega = 1$



2.1 Position, Orientation and Frames

- Orientation Matrix



$$^{A}R_B = \left[\,^{A}X_B \quad ^{A}Y_B \quad ^{A}Z_B\right]$$

$$R_A \cdot {}^{A}R_B = R_B$$

$$^{A}R_B = R_A^{-1} \cdot R_B = \begin{pmatrix} X_A^t \\ Y_A^t \\ Z_A^t \end{pmatrix} \left[X_B \quad Y_B \quad Z_B\right]$$

$$= \begin{pmatrix} X_B \cdot X_A & Y_B \cdot X_A & Z_B \cdot X_A \\ X_B \cdot Y_A & Y_B \cdot Y_A & Z_B \cdot Y_A \\ X_B \cdot Z_A & Y_B \cdot Z_A & Z_B \cdot Z_A \end{pmatrix}$$

2.1 Position, Orientation and Frames

- Remarks:
  1. Elements on $^{A}R_B$ are called directional cosine
  2. And $^{B}R_A = {}^{A}R_B^T$

- Coordinate Frame:
  - Referring to $\{A\}$, coordinate frame $\{B\}$ can be defined as:

$$\{B\} = \left\{{}^{A}R_B, {}^{A}P_{B\_origin}\right\}$$

2.2 Mapping Between Frames

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- Translated frame:

$$ {}^{A}P = {}^{B}P + {}^{A}P_{B\_origin} $$

- Rotated frame:

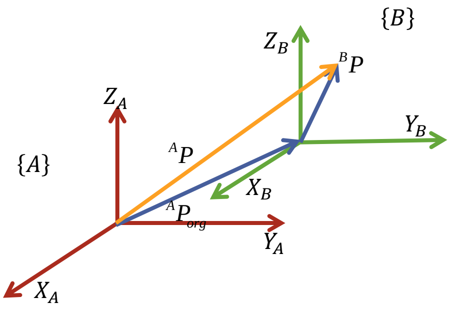
$$ {}^{A}P = {}^{A}R_{B}\,{}^{B}P $$

- General frame:

$$ {}^{A}P = {}^{A}R_{B}\,{}^{B}P + {}^{A}P_{B\_origin} $$

$$ \begin{pmatrix} {}^{A}P \\ 1 \end{pmatrix} = \left(\begin{array}{c|c} {}^{A}R_{B} & {}^{A}P_{B\_origin} \\ \hline 0 & 1 \end{array}\right) \begin{pmatrix} {}^{B}P \\ 1 \end{pmatrix} $$

4x4 matrix (Homogenous transformation)



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2.3 Transformation

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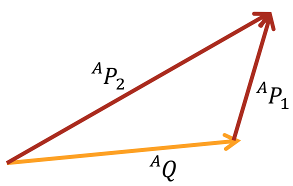
- Translation (commutative)

$$ {}^{A}P_2 = {}^{A}P_1 + {}^{A}Q $$

$$ {}^{A}P_2 = Trans(Q)\,{}^{A}P_1 $$

Prismatic revolute



$$ Trans(P) + Trans(Q) = Trans(Q) + Trans(P) $$

$$ Trans(Q) = \begin{pmatrix} 1 & 0 & 0 & q_x \\ 0 & 1 & 0 & q_y \\ 0 & 0 & 1 & q_z \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

Where $Q = (q_x q_y q_z)$

2.3 Transformation

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- Rotation (not commutative)

$${}^{A}P_2 = R_K(\theta)\,{}^{A}P_1 \qquad K: axis \quad \theta: angle$$

$$R_{k1}(\alpha) \bullet R_{k2}(\beta) \neq R_{k2}(\beta) \bullet R_{k1}(\alpha) \qquad except\ k_1 = k_2$$

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$x' = x$$
$$y' = \cos\theta \cdot y - \sin\theta \cdot z$$
$$z' = \sin\theta \cdot y + \cos\theta \cdot z$$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2.3 Transformation

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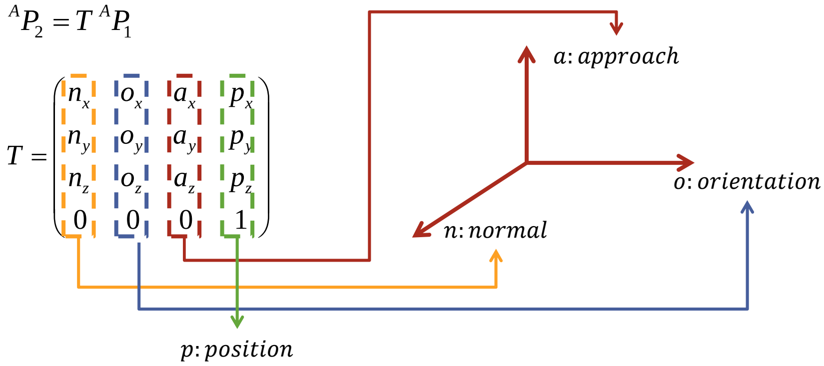
- General

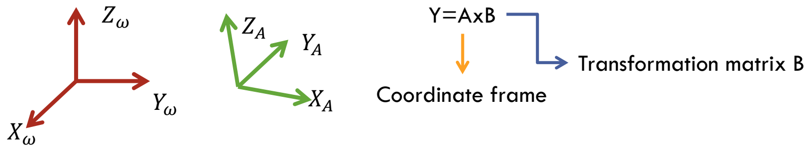
$${}^{A}P_2 = T\,{}^{A}P_1$$

$$T = \begin{pmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



2.3 Transformation

- Remarks:
  - **Y = A x B** (postmultiply), a transformation A by a second transformation B
  - → the transformation described by B is with respect to the *frame of A*.



  - **Y = B x A** (premultiply), a transformation A by a second transformation B
  - → the transformation described by B is with respect to the *base coordinate frame*.

2.3 Transformation

- Compound

$$^{B}P = {}^{B}T_C \bullet {}^{C}P$$

$$^{A}P = {}^{A}T_B \bullet {}^{B}P$$

$$^{A}P = {}^{A}T_B \bullet {}^{B}T_C \bullet {}^{C}P = {}^{A}T_C\, {}^{C}P$$

- Inversion

$$^{B}T_A = {}^{A}T_B^{-1}$$

$$T = \begin{bmatrix} n & o & a & p \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T^{-1} = \begin{pmatrix} n_x & n_y & n_z & -p \bullet n \\ o_x & o_y & o_z & -p \bullet o \\ a_x & a_y & a_z & -p \bullet a \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2.3 Transformation

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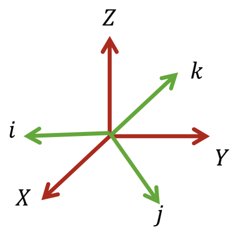
- General Rotation Transformation
  - Rotation about an arbitrary vector k from origin.
  - First find a coordinate frame C

$$C = \begin{pmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

k is the Z axis
i.e. $k = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$
$R_K(\theta) = R_{CZ}(\theta)$



Given a frame T, then we can find a frame X, s.t.

$$T = CX$$
$$X = C^{-1}T$$

2.3 Transformation

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- Rotating T about k is equivalent to rotating X around the Z axis of frame C

$$R_k(\theta)T = CR_Z(\theta)X = CR_Z(\theta)C^{-1}T$$

$$R_k(\theta) = CR_Z(\theta)C^{-1}$$

$$= \begin{pmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} n_x & n_y & n_z & 0 \\ o_x & o_y & o_z & 0 \\ a_x & a_y & a_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} K_x^2 \mathrm{vers}\,\theta + \cos\theta & K_x K_y vers\theta - K_z \sin\theta & K_x K_z vers\theta + K_y \sin\theta & 0 \\ K_x K_y vers\theta + K_z \sin\theta & K_y^2 \mathrm{vers}\,\theta + \cos\theta & K_y K_z vers\theta - K_x \sin\theta & 0 \\ K_x K_z vers\theta - K_y \sin\theta & K_y K_z vers\theta + K_x \sin\theta & K_z^2 \mathrm{vers}\,\theta + \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Where $vers\theta = 1 - cos\theta$ and $(k_x, k_y, k_z) = (a_x, a_y, a_z)$

2.4 Equivalent angle-axis of rotation

- Given a rotational transformation R

$$R=\begin{pmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Equate $R\ to\ R_k(\theta)$
- Define Trace (A)=sum of diagonal components of matrix A

$$Trace(R)=n_x+o_y+a_z+1$$

$$=K_x^2 vers\theta+\cos\theta+K_y^2 vers\theta+\cos\theta+K_z^2 vers\theta+\cos\theta+1=2+2\cos\theta$$

$$\therefore \cos\theta=\frac{1}{2}(n_x+o_y+a_z-1)$$

2.4 Equivalent angle-axis of rotation

$$o_z-a_y=2k_x s\theta$$

And

$$a_x-n_z=2k_y s\theta$$

$$n_y-o_x=2k_z s\theta$$

$$\therefore (o_z-a_y)^2+(a_x-n_z)^2+(n_y-o_x)^2=4\sin^2\theta$$

$$\sin\theta=\pm\frac{1}{2}\sqrt{(o_z-a_y)^2+(a_x-n_z)^2+(n_y-o_x)^2}$$

Note: why it is not preferred to solve q from: $k_x=\frac{o_z-a_y}{2\sin\theta}$

$$\tan\theta=\frac{\sin\theta}{\cos\theta}$$

Choose positive sign, then $0\le\theta\le 180°$

2.4 Equivalent angle-axis of rotation

- Another set:

$$\begin{cases} K_x^2 \operatorname{vers}\theta + \cos\theta = n_x \\ K_y^2 \operatorname{vers}\theta + \cos\theta = o_y \\ K_z^2 \operatorname{vers}\theta + \cos\theta = a_z \end{cases} \rightarrow \begin{cases} k_x = \pm\sqrt{\dfrac{n_x - \cos\theta}{1-\cos\theta}} \\ k_y = \pm\sqrt{\dfrac{o_y - \cos\theta}{1-\cos\theta}} \\ k_z = \pm\sqrt{\dfrac{a_z - \cos\theta}{1-\cos\theta}} \end{cases}$$

- From the largest element of n_x, o_y *and* a_z the largest of k_x, k_y *and* k_z can be determined

$$\text{If } n_x > \begin{cases} o_y \\ a_z \end{cases} \textit{then } k_x \textit{ is the largest} \quad \begin{aligned} n_y + o_x &= 2k_x k_y vers\theta \\ o_z + a_y &= 2k_y k_z vers\theta \\ n_z + a_x &= 2k_z k_x vers\theta \end{aligned} \quad \begin{aligned} \therefore k_y &= \frac{n_y + o_x}{2k_x vers\theta} \\ \therefore k_z &= \frac{n_z + a_x}{2k_x vers\theta} \end{aligned}$$

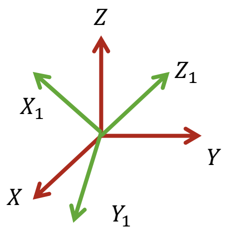
2.5 Three Angle Rotation: Euler angles, RPY Angles

- Euler Angels:

Roll Pitch Yaw

$$Euler(\phi, \theta, \psi) = R_Z(\phi) R_Y(\theta) R_Z(\psi) \quad \text{(z,y,z)}$$

a) Rotate ϕ about z-axis
b) Rotate θ about new y-axis
c) Rotate ψ about new z-axis



2.5 Three Angle Rotation: Euler angles, RPY Angles

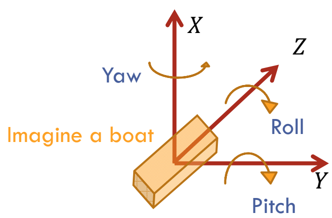
- Note: z−y−x or other combination are also possible.

$$=\begin{pmatrix}\cos\phi & -\sin\phi & 0 & 0\\ \sin\phi & \cos\phi & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} * \begin{pmatrix}\cos\theta & 0 & \sin\theta & 0\\ 0 & 1 & 0 & 0\\ -\sin\theta & 0 & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{pmatrix} * \begin{pmatrix}\cos\psi & -\sin\psi & 0 & 0\\ \sin\psi & \cos\psi & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

2.5 Three Angle Rotation: Euler angles, RPY Angles

$$RPY(\phi,\theta,\psi) = Rz(\phi)Ry(\theta)Rx(\psi)$$

a) Rotate ψ about x-axis
b) Rotate θ about y-axis
c) Rotate ϕ about z-axis



2.5 Three Angle Rotation: Euler angles, RPY Angles

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- Euler transformation solution

$Euler(\phi,\theta,\psi) = T$

$$= \begin{pmatrix} \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & -\cos\phi\cos\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\sin\theta & 0 \\ \sin\phi\cos\theta\cos\psi + \cos\phi\sin\psi & -\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\sin\theta & 0 \\ -\sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\frac{a_y}{a_x} = \frac{\sin\phi\sin\theta}{\cos\phi\sin\theta}$$

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{a_y}{a_x} = \frac{-a_y}{-a_x}$$

If $\theta = 0$ then $a_x = a_y = 0 \rightarrow$ *degenerate* because "pitch" can have no answer i.e. ϕ and ψ correspond to the same rotating axis.

It can be solved by setting $\phi = 0$ or the previous value or any value then solving for ψ

2.5 Three Angle Rotation: Euler angles, RPY Angles

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- For Euler z-y-z
  - a) Solution of ϕ:

    If $\theta \neq 0$ $\quad \Rightarrow \phi = \tan^{-1}\left[\frac{a_y}{a_x}\right]$ or $\Rightarrow \tan^{-1}\left[\frac{a_y}{a_x}\right] + 180^o$

  - b) Solution of θ:

    If $c\theta = a_z$ $\quad \Rightarrow s\theta = c\phi a_x + s\phi a_y = c\phi(c\phi s\theta) + s\phi(s\phi s\theta)$

    $$\theta = \tan^{-1}\left(\frac{s\theta}{c\theta}\right) = \tan^{-1}\left[\frac{c\phi a_x + s\phi a_y}{a_z}\right]$$

  - c) Solution of ψ:

$s\psi = -s\phi n_x + c\phi n_y = s\phi^2 s\psi + c\phi^2 s\psi$

$c\psi = -s\phi o_x + c\phi o_y$

$$\psi = \tan^{-1}\frac{s\psi}{c\psi} = \tan^{-1}\left\{\frac{-s\phi n_x + c\phi n_y}{-s\phi o_x + c\phi o_y}\right\}$$

2.5 Three Angle Rotation: Euler angles, RPY Angles

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- RPY transformation solution

$$RPY(\phi,\theta,\psi) = Rz(\phi)Ry(\theta)Rx(\psi)$$

$$= \begin{bmatrix} \cos\phi\cos\theta & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & 0 \\ \sin\phi\cos\theta & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & 0 \\ -\sin\theta & \cos\theta\sin\psi & \cos\theta\cos\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.5 Three Angle Rotation: Euler angles, RPY Angles

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- RPY transformation solution

When $\theta = \pm 90° \rightarrow \cos\theta = 0 \Rightarrow$ Degeneracy.

a) When $\theta \neq \pm 90^{o}$ $\phi = \tan^{-1}\dfrac{n_y}{n_x}$ or $\tan^{-1}\dfrac{n_y}{n_x} + 180^{\circ}$

b) $$\left.\begin{aligned} \sin\theta &= -n_z \\ \cos\theta &= \cos\phi n_x + \sin\phi n_y \end{aligned}\right\} \Rightarrow \theta = \tan^{-1}\left(\frac{-n_z}{\cos\phi n_x + \sin\phi n_y}\right)$$

c) Solution ψ:

$$\left.\begin{aligned} \sin\psi &= \sin\phi a_x - \cos\phi a_y \\ \cos\psi &= -\sin\phi o_x + \cos\phi o_y \end{aligned}\right\} \Rightarrow \psi = \tan^{-1}\left(\frac{\sin\phi a_x - \cos\phi a_y}{-\sin\phi o_x + \cos\phi o_y}\right)$$

2.6 Specification of Position

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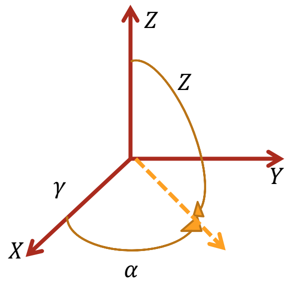
- Cylindrical Coordinates:
  - $cyl(z,\alpha,\gamma) = trans(0,0,z) * R_z(\alpha) * Trans(\gamma,0,0)$

1) Translate γ along x-axis
2) Rotate α about z-axis
3) Translate z along z-axis

$$= \begin{pmatrix} c\alpha & -s\alpha & 0 & \gamma c\alpha \\ s\alpha & c\alpha & 0 & \gamma s\alpha \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad P = \begin{pmatrix} \gamma c\alpha \\ \gamma s\alpha \\ Z \\ 1 \end{pmatrix} \begin{matrix} x \\ y \\ z \\ \omega \end{matrix}$$



2.6 Specification of Position

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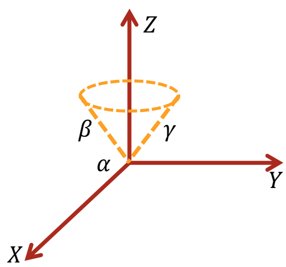
- Spherical Coordinates:
  - $S(\alpha,\beta,\gamma) = R_z(\alpha) * R_y(\beta) * Trans(0,0,\gamma)$

1) Translate γ along z-axis
2) Rotate β about y-axis
3) Rotate α about z-axis

$$= \begin{pmatrix} c\alpha c\beta & -s\alpha & c\alpha s\beta & \gamma c\alpha s\beta \\ s\alpha c\beta & c\alpha & s\alpha s\beta & \gamma s\alpha s\beta \\ -s\beta & 0 & c\beta & \gamma c\beta \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad P = \begin{pmatrix} \gamma c\alpha s\beta \\ \gamma s\alpha s\beta \\ \gamma c\beta \\ 1 \end{pmatrix} \begin{matrix} x \\ y \\ z \\ \omega \end{matrix}$$



2.6 Specification of Position

- Summary:

Translation	Rotation
p	$n\ o\ a$
$Cartesian(x, y, z)$	$R_k(\theta)$
$Cyl(z, \alpha, \gamma)$	$Euler(\phi, \theta, \psi)$
$Sph(\alpha, \beta, \gamma)$	$RPY(\phi, \theta, \psi)$