Finding #(2p) from a Product of Sines

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*The American Mathematical Monthly* is currently published by Mathematical Association of America.
Finding $\zeta(2p)$ From a Product of Sines

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The zeta function $\zeta(z)$ given by the series $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ (valid when $\text{Re}(z) > 1$) was first evaluated in closed form by Euler [5] for $z$ a positive even integer $2p$. The result is

$$\zeta(2p) = \sum_{n=1}^{\infty} \frac{1}{n^{2p}} = \frac{(-1)^{p+1}2^{2p-1}B_{2p}}{(2p)!} \pi^{2p}. \quad (1)$$

Here the numbers $B_n$ are called Bernoulli numbers, and they are all rational. The first few are

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \ldots.$$ 

while $B_3 = B_5 = B_7 = \ldots = 0$. These can all be calculated recursively by starting with $B_0 = 1$ and using the identity $B_n = \sum_{k=0}^{n} \binom{n}{k} B_k$ for $n = 2, 3, 4, \ldots$. When $n = 2$, this gives $B_2 = -1/2$. (See Knopp [6, p. 183] for an equivalent formula.) Several additional methods of deriving (1) have been given since Euler’s time, some of which are found in [1], [3], [4], and [6]. We present a method here that we were unable to locate in the literature.

We begin our evaluation of $\zeta(2p)$ by considering the following two lemmas.

**Lemma 1.** If $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series of complex numbers, then the infinite product $\prod_{n=1}^{\infty} (1 + a_n z)$ is an entire function of $z$ whose power series expansion about the origin begins as follows:

$$\prod_{n=1}^{\infty} (1 + a_n z) = 1 + \left( \sum_{n=1}^{\infty} a_n \right) z + \cdots.$$ 

This lemma is an immediate consequence of [6, Example 1, p. 439], where explicit formulas are also given for the coefficients of $z^2, z^3, \ldots$ (Euler himself used this lemma but he required the coefficients of the higher powers of $z$ as well. In this paper, we need only the constant and first-order terms of the expansion.)
The next lemma is the well-known factorization of $1 - t^p$ in terms of $p$th roots of unity.

**Lemma 2.** If $\omega = e^{\pi i/p}$, where $p$ is a positive integer, the following algebraic identity is valid for all $t$: $\prod_{k=0}^{p-1} (1 - \omega^k t) = 1 - t^p$.

When $t = z^2/(\pi^2 n^2)$, Lemma 2 gives

$$\prod_{k=0}^{p-1} \left(1 - \frac{(\omega^k z)^2}{\pi^2 n^2}\right) = 1 - \frac{z^{2p}}{\pi^2 p n^{2p}}. \tag{2}$$

We next use the infinite product representation

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 n^2}\right)$$

with $z$ replaced by $\omega^k z$ and form the finite product of sines

$$\prod_{k=0}^{p-1} \sin(\omega^k z) = \omega^{p(p-1)/2} z^p \prod_{n=1}^{\infty} \prod_{k=0}^{p-1} \left(1 - \frac{(\omega^k z)^2}{\pi^2 n^2}\right)$$

$$= \omega^{p(p-1)/2} z^p \prod_{n=1}^{\infty} \left(1 - \frac{z^{2p}}{\pi^2 n^{2p}}\right),$$

where in the last step we invoke (2). Now define

$$g(z) = \omega^{p(1-p)/2} \prod_{k=0}^{p-1} \sin(\omega^k z) = z^p \prod_{n=1}^{\infty} \left(1 - \frac{z^{2p}}{\pi^2 n^{2p}}\right)$$

and apply Lemma 1 to obtain the power series expansion

$$g(z) = \omega^{p(1-p)/2} \prod_{k=0}^{p-1} \sin(\omega^k z) = z^p - \frac{\zeta(2p)}{\pi^2 p} z^{3p} + \cdots. \tag{3}$$

To find $\zeta(2p)$ from (3), we could replace each sine function with its Taylor series, multiply these series, and then equate coefficients of $z^{3p}$. However, to derive (1), it is easier if we first take the derivative of $g(z)$. From (3) we get

$$g'(z) = p z^{p-1} - 3 p \frac{\zeta(2p)}{\pi^2 p} z^{3p-1} + \cdots. \tag{4}$$

Another form of expansion (4) can be obtained by logarithmic differentiation of the finite product defining $g(z)$. This leads to

$$\frac{g'(z)}{g(z)} = \sum_{k=0}^{p-1} \omega^k \cot(\omega^k z),$$

hence to
\[ g'(z) = g(z) \sum_{k=0}^{p-1} \omega^k \cot(\omega^k z) = \left( z^p - \frac{\zeta(2p)}{\pi^{2p}} z^{2p} + \cdots \right) \sum_{k=0}^{p-1} \omega^k \cot(\omega^k z). \quad (5) \]

To evaluate \( \zeta(2p) \) we equate the coefficient of \( z^{3p-1} \) in (4) with that in (5). Contributions to this coefficient in (5) come from two sources arising from the Laurent expansion of the sum of cotangents, namely, from the coefficient of \( z^{-1} \) and from the coefficient of \( z^{2p-1} \). Because

\[ \cot z = \frac{1}{z} + \sum_{r=1}^{\infty} c_r z^{2r-1}, \]

where \( c_r = (-1)^r 2^{2r} B_{2r}/(2r)! \),

\[ \omega^k \cot(\omega^k z) = \frac{1}{z} + \omega^k \sum_{r=1}^{\infty} c_r (\omega^k z)^{2r-1} = \frac{1}{z} + \sum_{r=1}^{\infty} c_r \omega^{2rk} z^{2r-1}. \]

When this is summed over \( k \) the total contribution from \( z^{-1} \) is \( p \), while that from \( z^{2p-1} \) is \( pc_p \), because \( \omega^{2p} = 1 \). Equating the coefficient of \( z^{3p-1} \) in (4) with the corresponding one in (5), we find that

\[-3p \frac{\zeta(2p)}{\pi^{2p}} = -p \frac{\zeta(2p)}{\pi^{2p}} + pc_p.\]

This gives \( \zeta(2p) = -c_p \pi^{2p}/2 = (-1)^{p+1} 2^{2p-1} B_{2p}/(2p)! \), as required. \( \blacksquare \)

ACKNOWLEDGEMENT. The author wishes to thank the referee for considerably improving the exposition of this note.

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A Very Simple and Elementary Proof of a Theorem of Ingelstam

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1. INTRODUCTION. The aim of this note is to give a very simple and elementary proof of the following interesting theorem due to Ingelstam [4].

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