An Elementary Proof of Euler's Formula for #\((2m)\)

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An Elementary Proof of Euler’s Formula for $\zeta(2m)$

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Let $\zeta(s)$ be the Riemann zeta function. Euler proved the fascinating formula

$$\zeta(2m) = \frac{(-1)^{m-1} 2^{2m-1} \pi^{2m}}{(2m)!} B_{2m}$$

(1)

for any positive integer $m$, where $\{B_n\}$ is the sequence of Bernoulli numbers defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

Various proofs of (1) have been obtained (see, for example, [3]). In an issue of this MONTHLY [5], Williams gave an elementary proof of (1). In [1], Boo Rim Choe produced another elementary proof of (1) in the case $m = 1$.

In this note, we present a new elementary proof of (1). The author has introduced a method to evaluate $\zeta(2m)$ by the calculation of $q$-series (see [4]). The method here is more simple and direct.

For $d > 0$ and $u$ in $[1, 1 + d]$, we let

$$\frac{2e^t}{e^t + u} = \sum_{n=0}^{\infty} \varepsilon_n(u) \frac{t^n}{n!} \quad (|t| < \pi).$$

(2)

Since $u \geq 1$, we have

$$\lim_{n \to \infty} \inf_{u > 1} \left( \frac{|\varepsilon_n(u)|}{n!} \right)^{-1/n} \geq \pi.$$

(3)

Corresponding to $\varepsilon_n(u)$, we consider $\phi(s; u) = \sum_{n \geq 1} (-u)^{-n} \frac{s^n}{n!}$ for real $s$. As is well known, $\phi(s; u)$ is convergent for $s > 0$ when $u = 1$ and is convergent for any real $s$ when $u > 1$. Furthermore, we see that $\phi(s; 1) = (2^{1-s} - 1)\zeta(s)$. Assume that $u > 1$. Since the left-hand side of (2) expands to $-2 \sum_{n \geq 1} (-u)^{-n} e^{nt}$, we have $\varepsilon_n(u) = -2\phi(-m; u)$ for any nonnegative integer $m$. For any positive integer $k$, we have

$$0 = i \sum_{n=1}^{\infty} \frac{(-u)^{-n} \sin(n\pi)}{n^{2k+1}} = \sum_{j=0}^{\infty} \phi(2k - 2j; u) \frac{(i\pi)^{2j+1}}{(2j+1)!}$$

$$= \sum_{j=0}^{k-1} \phi(2k - 2j; u) \frac{(i\pi)^{2j+1}}{(2j+1)!} - \frac{1}{2} \sum_{m=0}^{\infty} \varepsilon_{2m}(u) \frac{(i\pi)^{2m+2k+1}}{(2m+2k+1)!}.$$  

(4)

Because $k \geq 1$, we infer with the aid of (3) that the right-hand side of (4) is uniformly convergent with respect to $u$ on $(1, 1 + d]$. Hence we can let $u \to 1$ in the right-hand side of (4). Note that $\varepsilon_n(1) = E_n(1)$, where $E_n(X)$ is the $n$th Euler polynomial.
(see [2]). In particular, \( e_{2m}(1) = E_{2m}(1) = 0 \) for any positive integer \( m \), and \( e_0(1) = 1 \). Hence

\[
0 = \sum_{j=0}^{k-1} \phi(2k-2j; 1) \frac{(i\pi)^{2j+1}}{(2j+1)!} - \frac{(i\pi)^{2k+1}}{2(2k+1)!}.
\]  

(5)

For simplicity, we define

\[
A_{2m} = \phi(2m; 1) \frac{(2m)!}{(i\pi)^{2m}} = (2^{1-2m} - 1)\zeta(2m) \frac{(2m)!}{(i\pi)^{2m}}
\]  

(6)

for \( m = 1, 2, \ldots \) and \( A_0 = -1/2 \). Then (5) states that

\[
\sum_{j=0}^{k} \binom{2k+1}{2j+1} A_{2k-2j} = 0
\]

when \( k \geq 1 \). Since \( A_0 = -1/2 \), we obtain

\[
-\frac{t}{2} = \sum_{k=0}^{\infty} \left( \sum_{j=0}^{k} \binom{2k+1}{2j+1} A_{2k-2j} \right) \frac{t^{2k+1}}{(2k+1)!}
\]

\[
= \left( \sum_{m=0}^{\infty} A_{2m} \frac{t^{2m}}{(2m)!} \right) \frac{e^t - e^{-t}}{2}.
\]

We can easily check that

\[
\frac{2t}{e^t - e^{-t}} = \frac{2te^t}{e^{2t} - 1} = \sum_{m=0}^{\infty} (2 - 2^{2m}) B_{2m} \frac{t^{2m}}{(2m)!}
\]

so we have \( A_{2m} = (2^{2m-1} - 1)B_{2m} \) for any nonnegative integer \( m \). In view of (6), we obtain (1).

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