9 Transmission Lines

9.1 Introduction

1. EM wave in free space and in waveguides or transmission lines

2. Examples of transmission lines

   - two-wire lines
   - coaxial cable
   - metal interconnection in PCB or IC

9.3 General Transmission Line Equations

1. Electric circuit: $\ell \ll \lambda$, no current/voltage variation in lumped elements

   Transmission line: $\ell \gg, \geq \lambda$, distributed circuit elements

2. Distributed model of transmission lines

   Consider a differential length Δz of a general two conductor transmission line (parallel plate, coaxial cables, etc.) described by

$$\begin{cases} R = \text{resistance per unit length of the conductors } (\Omega/\text{m}) \\ L = \text{inductance per unit length of the line (H/m)} \\ G = \text{conductance per unit length of the dielectric medium (S/m)} \\ C = \text{capacitance per unit length of the line (F/m)} \end{cases}$$

   (see Tables 9-1 and 9-2)

3. Applying Kirchhoff's voltage law, we have

4. Applying Kirchhoff's current law at node N, we have

((9-31) and (9-33) are the general transmission line equations (telegrapher's equations)

5. In phasor form,

$$\begin{cases} v(z,t) &= \Re[V(z)e^{j\omega t}] \\ i(z,t) &= \Re[I(z)e^{j\omega t}] \end{cases}$$

$$\Rightarrow \begin{cases} -\dfrac{dV(z)}{dz} &= (R+j\omega L)I(z) \\ -\dfrac{dI(z)}{dz} &= (G+j\omega C)V(z) \end{cases} \quad \text{(9-35)}$$

- time harmonic transmission line equation
- In circuit theory, Fourier transform reduces an ordinary differential equation to an algebraic equation.
  In EM and transmission line theory, Fourier transform reduces a partial differential equation to an ordinary differential equation.

9.3.1 Wave on an Infinite Transmission Line

1. Decoupled wave equations:

$$\text{where} \begin{cases} \gamma &= \alpha + j\beta = \sqrt{(R+j\omega L)(G+j\omega C)} = \text{ propagation constant} \\ \alpha &= \text{attenuation constant} \\ \beta &= \text{phase constant} \end{cases}$$

2. Solution of the wave equations:

For an inifinte line, only $V^+(z)$ and $I^+(z)$ exist (no reflection)

$$\begin{cases} V(z) &= V^+(z) = V_0^+ e^{-\gamma z} \\ I(z) &= I^+(z) = I_0^+ e^{-\gamma z} \\ Z_0 &\triangleq \dfrac{V(z)}{I(z)} = \dfrac{V_0^+}{I_0^+} = \text{constant} \end{cases}$$

$\Rightarrow Z_0 =$ characteristic impedance of the line (independent of length)

Ex 9-2: Analogy between waves on a transmission line and in a dielectric medium

- For a uniform plane wave with E_x and H_y only, the Maxwell's equations can be reduced to

$$\begin{cases} \nabla \times \mathbf{E} &= -j\omega(\mu' - j\mu'')\mathbf{H} \\ \nabla \times \mathbf{H} &= j\omega(\epsilon' - j\epsilon'')\mathbf{E} \end{cases} \Rightarrow \begin{cases} -\dfrac{dE_x(z)}{dz} &= (\omega\mu'' + j\omega\mu')H_y(z) \\ -\dfrac{dH_y(z)}{dz} &= (\omega\epsilon'' + j\omega\epsilon')E_x(z) \end{cases}$$

$$\Rightarrow \begin{cases} E_x &\leftrightarrow V \\ H_y &\leftrightarrow I \end{cases} \text{ and } \begin{cases} R &\leftrightarrow \omega\mu'' \\ L &\leftrightarrow \mu' \\ G &\leftrightarrow \omega\epsilon'' \\ C &\leftrightarrow \epsilon' \end{cases}$$

$$\gamma = \alpha + j\beta = \omega\sqrt{\mu\epsilon} = \sqrt{(\omega\mu'' + j\omega\mu')(\omega\epsilon'' + j\omega\epsilon')}$$

$$Z_0 \leftrightarrow \eta_c = \sqrt{\frac{\mu'' + j\mu'}{\epsilon'' + j\epsilon'}}$$

- Reflection from interface or discontinuity

3. Characteristic impedance and propagation constant

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$$

(a) lossless lines ($R = 0, G = 0$)

$\gamma = j\omega\sqrt{LC}, \alpha = 0, \beta = \omega\sqrt{LC}$

$u_p = \omega/\beta = 1/\sqrt{LC}$

$$Z_0 = R_0 + jX_0 = \sqrt{L/C} \Rightarrow \begin{cases} R_0 = \sqrt{L/C} = \text{constant} \\ X_0 = 0 \end{cases}$$

(b) low loss lines ($R \ll \omega L, G \ll \omega C$)

$$\Rightarrow \alpha \approx \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right)$$

$\beta \approx \omega\sqrt{LC}$

$u_p = \omega/\beta \approx 1/\sqrt{LC}$ (approximately constant)

$$Z_0 = R_0 + jX_0 \approx \sqrt{\frac{L}{C}}\left(1 + \frac{1}{j2\omega}\left(\frac{R}{L} - \frac{G}{C}\right)\right)$$

$$\Rightarrow R_0 \approx \sqrt{\frac{L}{C}},$$

$$X_0 \approx \sqrt{\frac{L}{C}}\frac{1}{2\omega}\left(\frac{R}{L} - \frac{G}{C}\right) \rightarrow 0 \text{ for low loss lines}$$

(c) distortionless (dispersionless) lines ($\frac{R}{L} = \frac{G}{C}$)

$$\Rightarrow \alpha = R\sqrt{\frac{C}{L}} = \text{ constant}$$

$\beta = \omega\sqrt{LC}$

$u_p = \omega/\beta = 1/\sqrt{LC} =$ constant $\Rightarrow$ no distortion

$Z_0 = \cdots = \sqrt{L/C}$

$\Rightarrow R_0 = \sqrt{L/C}, X_0 = 0$

Ex 9-3: 50Ω Distortionless lines

9.4 Wave on Finite Transmission Lines

1. For a general transmission line,

$$\begin{cases} V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z} \\ I(z) = \underbrace{I_0^+ e^{-\gamma z}}_{+z \text{ wave}} + \underbrace{I_0^- e^{+\gamma z}}_{-z \text{ wave}} \end{cases}$$

$$\frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = Z_0 = \text{characteristic impedance}$$

2. Now consider a finite line (Z_0) terminated with a load impedance Z_L,

   at $z = \ell$, V and I must satisfy

   (a) If $Z_L = Z_0$ (matched line) $\Rightarrow$ no backward (reflected) wave

   (b) If $Z_L \neq Z_0$ (unmatched line) $\Rightarrow$ a reflected wave must exit in order to satisfy the above boundary condition

3. $V(z)$ and $I(z)$ can be expressed in terms of V_i, I_i or V_L, I_L:

$$\begin{cases} V(\ell) = V_L = V_0^+ e^{-\gamma \ell} + V_0^- e^{+\gamma \ell} \\ I(\ell) = I_L = \frac{V_0^+}{Z_0} e^{-\gamma \ell} - \frac{V_0^-}{Z_0} e^{\gamma \ell} \end{cases}$$

$$\Rightarrow \begin{cases} V_0^+ = \frac{1}{2}(V_L + I_L Z_0) e^{\gamma \ell} \\ V_0^- = \frac{1}{2}(V_L - I_L Z_0) e^{-\gamma \ell} \end{cases}$$

   Let $z' = \ell - z =$ distance backward from the load

$$\begin{cases} V(z') = \frac{I_L}{2}[(Z_L + Z_0) e^{\gamma z'} + (Z_L - Z_0) e^{-\gamma z'}] \\ I(z') = \frac{I_L}{2Z_0}[\underbrace{(Z_L + Z_0) e^{\gamma z'}}_{-z' \text{ wave}} - \underbrace{(Z_L - Z_0) e^{-\gamma z'}}_{+z' \text{ wave}}] \end{cases}$$

Or, $V(z') = \frac{I_L}{2}[Z_L(e^{\gamma z'} + e^{-\gamma z'}) + Z_0(e^{\gamma z'} - e^{-\gamma z'})]$

$\Rightarrow$

$\Rightarrow Z(z') = \frac{V(z')}{I(z')}$ = impedance at z' looking toward the load

4. At the source, $z' = \ell$,

   Z_i = input impedance seen by the generator

   $\Rightarrow$ equivalent circuit

5. Average power delivered by the generator to the line:

   $$(P_{\text{av}})_i = \frac{1}{2}\Re[V_i I_i^*]|_{z=0,z'=\ell}$$

   Average power delivered to the load

   $$(P_{\text{av}})_L = \frac{1}{2}\Re[V_L I_L^*]|_{z=\ell,z'=0} = \frac{1}{2}\left|\frac{V_L}{Z_L}\right|^2 R_L = \frac{1}{2}I_L^2 R_L$$

   (for lossless lines, $(P_{\text{av}})_i = (P_{\text{av}})_L$)

6. If $Z_L = Z_0$ (matched),

   $\rightarrow$ (9-98) is reduced to

   $$\begin{cases} V(z) = V_i e^{-\gamma z} \\ I(z) = V_i e^{-\gamma z} \end{cases}$$

   $\Rightarrow$ no reflection wave, similar to infinite line

Ex 9-5: 50Ω lossless matched line

(a) instantaneous V and I at arbitrary locations

(b) instantaneous V and I at load

(c) average power to the load

9.4.1 Transmission Lines as Circuit Elements

1. At UHF, f = 300 MHz ∼ 3 GHz, λ=1 m ∼ 0.1 m. Lumped elements are difficult to make (stray field effect).
   ⇒ finite transmission line can be used as circuit elements.

2. Consider a lossless transmission line segment,

$$Z(z') = Z_0 \frac{Z_L + Z_0 \tanh \gamma z'}{Z_0 + Z_L \tanh \gamma z'}$$

(a) $Z_L \to \infty$ (open circuit termination)

(Note: It is difficult to obtain $Z_L \to \infty$ due to field coupling and radiation.)

(b) $Z_L = 0$ (short circuit termination)

(c) Quarter wave section ($\ell = \lambda/4, \beta\ell = \pi/2$)

or $\ell = (2n-1)\lambda/4, n = 1, 2, 3..., \Rightarrow \tan\beta\ell = \pm\infty$

$\Rightarrow$

(d) Half wave section ($\ell = \lambda/2, \beta\ell = \pi$)

or $\ell = n\lambda/2, n = 1, 2, 3..., \Rightarrow \tan\beta\ell = 0$

$\Rightarrow Z_i = Z_L$ (why?)

3. By measuring $Z_{io} = Z_0 \coth\gamma\ell$ and $Z_{is} = Z_0 \tanh\gamma\ell$, we can determine

$$Z_0 = \sqrt{Z_{io}Z_{is}} \quad (\Omega)$$
$$\gamma = \frac{1}{\ell}\tanh^{-1}\sqrt{\frac{Z_{is}}{Z_{io}}} \quad (\mathrm{m}^{-1})$$

Ex 9-6: Open- and short-circuited impedance

(a)

(b)

(c)

9.4.2 Resistive Termination

1. For $z' = \ell - z$,

$$V(z') = \frac{I_L}{2}[\underbrace{(Z_L + Z_0)e^{\gamma z'}}_{-z' \text{ incident wave}} + \underbrace{(Z_L - Z_0)e^{-\gamma z'}}_{+z' \text{ reflected wave}}]$$

Similarly,

$$I(z') = \frac{I_L}{2Z_0}(Z_L + Z_0)e^{\gamma z'}(1 - \Gamma e^{-2\gamma z'})$$

2. For lossless lines, $\gamma = j\beta, Z_0 = R_0$,
$$\begin{cases} V(z') &= \frac{I_L}{2}(Z_L + R_0)e^{j\beta z'}(1 + |\Gamma|e^{j(\theta_\Gamma - 2\beta z')}) \\ I(z') &= \frac{I_L}{2R_0}(Z_L + R_0)e^{j\beta z'}(1 - |\Gamma|e^{j(\theta_\Gamma - 2\beta z')}) \end{cases} \quad (9-135)$$
Or from (9-100, p. 9-6) and $V_L = I_L Z_L, Z_L = R_L$ (purely resistive load)
$$\begin{cases} |V(z')| &= V_L\sqrt{\cos^2\beta z' + (R_0/R_L)^2\sin^2\beta z'} \\ |I(z')| &= I_L\sqrt{\cos^2\beta z' + (R_0/R_L)^2\sin^2\beta z'} \end{cases} \quad (9-137)$$
3. Sanding wave on transimission lines
   (a) For $R_L = R_0$, or, in general, $Z_L = Z_0$, there is no reflected wave ($\Gamma = 0$) and $|V(z')| = V_L =$ constant

   (b) For $R_L \neq R_0$, or, in general, $Z_L \neq Z_0$, there is a reflected wave ($\Gamma \neq 0$) and $|V(z')| \neq$ constant due to the interference of the two waves

   (Note: the period of the standing wave pattern is $\lambda/2$)
   (c) The standing wave ratio is defined as

   For a lossless line,
   - $Z_L = Z_0$ (matched) $\Rightarrow \Gamma = 0, S = 1$
   - $Z_L = 0$ (short circuited) $\Rightarrow \Gamma = -1, S \to \infty$
   - $Z_L = \infty$ (open circuited) $\Rightarrow \Gamma = 1, S \to \infty$
4. max and min of V and I on the transmission line for general load
From (9-135),

$|V_{\max}|, |I_{\min}|$ occurs at $\theta_\Gamma - 2\beta z' = -2n\pi, n = 0, 1, 2, ...$
$|V_{\min}|, |I_{\max}|$ occurs at $\theta_\Gamma - 2\beta z' = -(2n+1)\pi, n = 1, 2, ...$

For resistive termination on a lossless line, both Z_L and Z_0 are real,

(1) $R_L > R_0, \theta_\Gamma = 0$

(2) $R_L < R_0, \theta_\Gamma = \pi$

9.4.3 Arbitrary Termination (see Smith Chart)

9.4.4 Transmission Line Circuit

1. To express $V(z')$ in terms of V_g, let us consider the multiple reflections at $z = 0$ and $z = \ell$:

$$\begin{aligned} V(z') &= \sum \text{all waves traveling in the line} \\ &= V_1^+ + V_1^- + V_2^+ + V_2^- + ... \\ &= \end{aligned}$$

If $Z_L = Z_0 \Rightarrow$ only V_1^+ exists ($\Gamma = 0$)

If $Z_L \neq Z_0$ but $Z_g = Z_0 \rightarrow \Gamma \neq 0, \Gamma_g = 0 \Rightarrow$ only V_1^+ and V_1^- exist

Ex 9-10: 50 Ω lossless air line

9.6 Smith Chart

1. For a lossless line, the reflection coefficient Γ is

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{Z_L - R_0}{Z_L + R_0} = |\Gamma| e^{j\theta_\Gamma}$$

2. If the load impedance is normalized to the line impedance R_0,

   It can be shown that $\begin{cases} r &= \dfrac{1 - \Gamma_r^2 - \Gamma_i^2}{(1-\Gamma_r)^2 + \Gamma_i^2} \\ x &= \dfrac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^2} \end{cases}$

3. The real part of the normailzed load impedance r can be plotted in the complex Γ plane. It can be shown that

   $\Rightarrow$ circle centered at $(\Gamma_r, \Gamma_i) = (\frac{r}{1+r}, 0)$ with radius $= \frac{1}{1+r}$

   Similarly, the imaginary part x satisfies

   $\Rightarrow$ circle centered at $(1, \frac{1}{x})$ with radius $= \frac{1}{x}$

(a) Constant-r circles (Eq. 9-188):

i. centered on the Γ_r axis ($\Gamma_i = 0$)

ii. largest circle: $r = 0$ centered at (0, 0)

iii. for $r \to \infty$, r-circle $\to$ (1, 0) (open circuit)

iv. all r-circles pass through (1, 0)

(b) Constant-x circles (Eq. 9-189):

i. centered on the $\Gamma_r = 1$ line

ii. for $x = 0 \Rightarrow \Gamma_r$ axis

iii. for $x = 0 \to \infty$, x-circle $\to$ (1, 0) (open circuit)

iv. all x-circles pass through (1, 0)

4. Smith Chart:

(a) r- and x-circles are plotted in $\Gamma_r - \Gamma_i$ plane for $|\Gamma| \leq 1$;

(b) Intersection of a r- and a x-circle represents a normalized load impedance $z_L = r + jx$.

Ex: In Fig. 9-30, the point P is found to be $(r, x) = (1.7, 0.6)$. Therefore, the normalized impedance is

and the real load impedance is

Some interesting points on the chart:

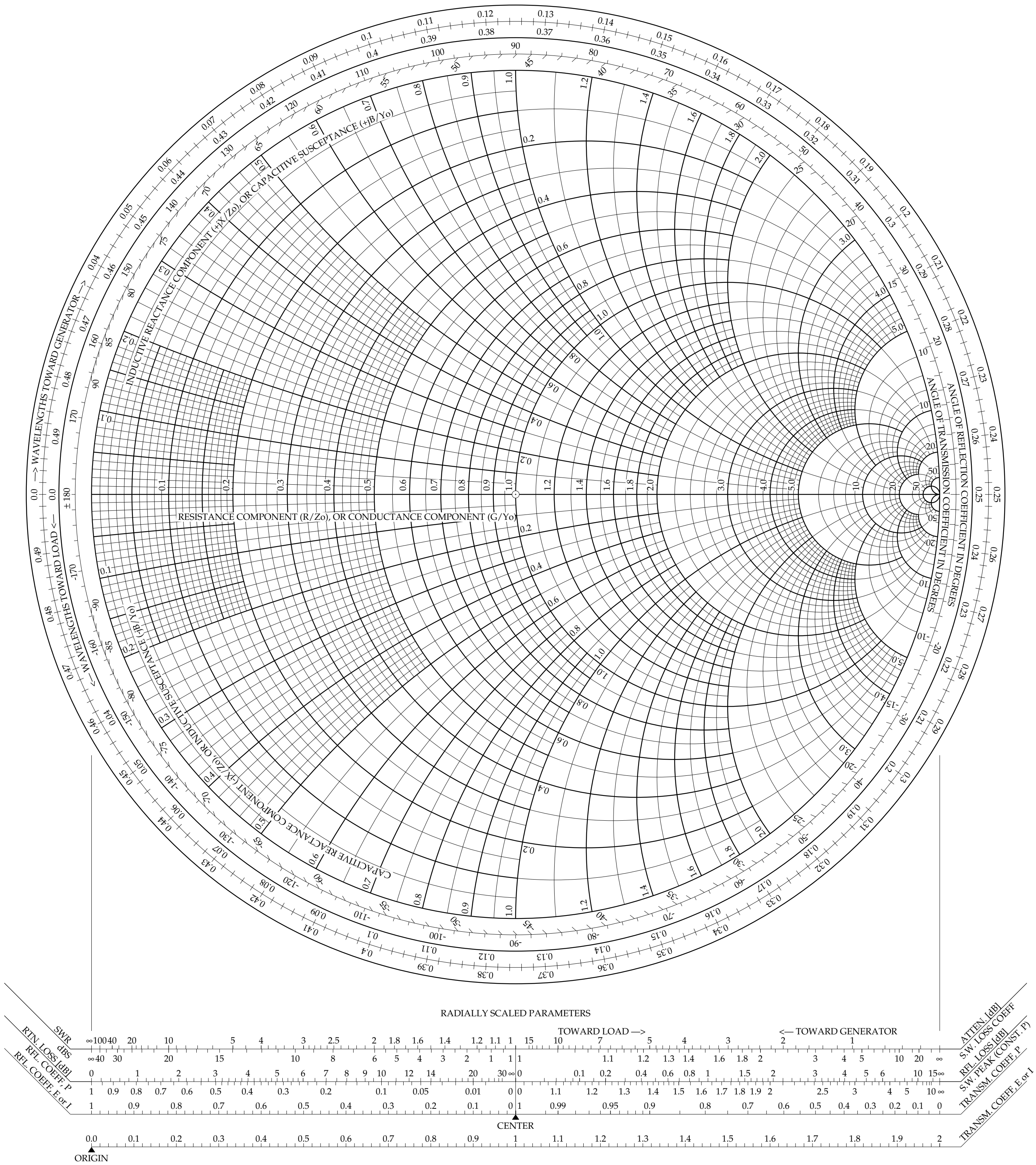
$P_{SC} : (r, x) =$

$P_{OC} : (r, x) =$

(c) Once the point representing the normalized impedance is located, the reflection coefficient $\Gamma = |\Gamma| e^{j\theta_\Gamma}$ can be found from the chart, and vise versa.

The Complete Smith Chart

Black Magic Design



(Source: Wikipedia)

(d) Constant-$|\Gamma|$ circles:

i. centered at (0, 0) and $0 \leq |\text{radius}| \leq 1$

ii. P_M, P_m:

$\Rightarrow$ SWR=the r-circle passing through P_M

5. Input impedance calculation using Smith Chart

$$Z_i(z') = \frac{V(z')}{I(z')} = Z_0 \frac{1 + \Gamma e^{-j2\beta z'}}{1 - \Gamma e^{-j2\beta z'}} \text{ (Eq. 9-133, } \gamma = \text{j}\beta)$$

$\Rightarrow$ on the Γ-plane (Smith chart), $z_i(z')$ is located on the $|\Gamma|$-circle with phase angle $\phi = \theta_\Gamma - 2\beta'$

But $|\Gamma|$ and S are independent of z'

$\Rightarrow z_i(z')$ can be found from proper rotation of z_L on the $|\Gamma|$-circle:

Given $z_L = z_i(0)$,

(a) find $\Gamma = |\Gamma| e^{j\theta_\Gamma}$;

(b) rotate the Γ point clockwise along the $|\Gamma|$-circle by an angle of $2\beta z' = 4\pi z'/\lambda$ to Γ' (subtraction of phase)

(c) $\Gamma' = |\Gamma| e^{j(\theta_\Gamma - 2\beta z')} = |\Gamma| e^{j\phi}$ corresponds to the input impedance z_i at z'

The Complete Smith Chart

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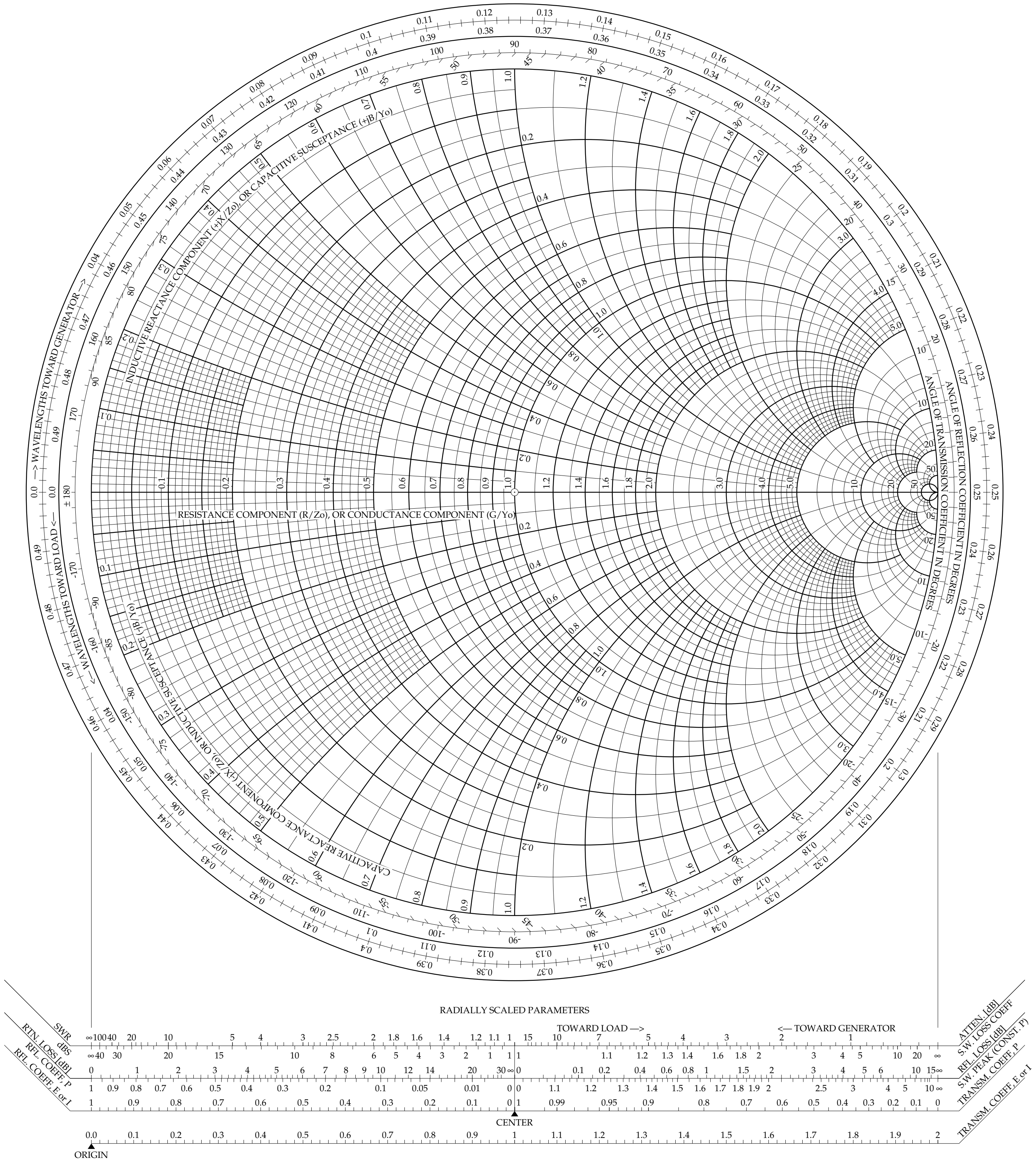

(Source: Wikipedia)

Ex 9-13:

The Complete Smith Chart

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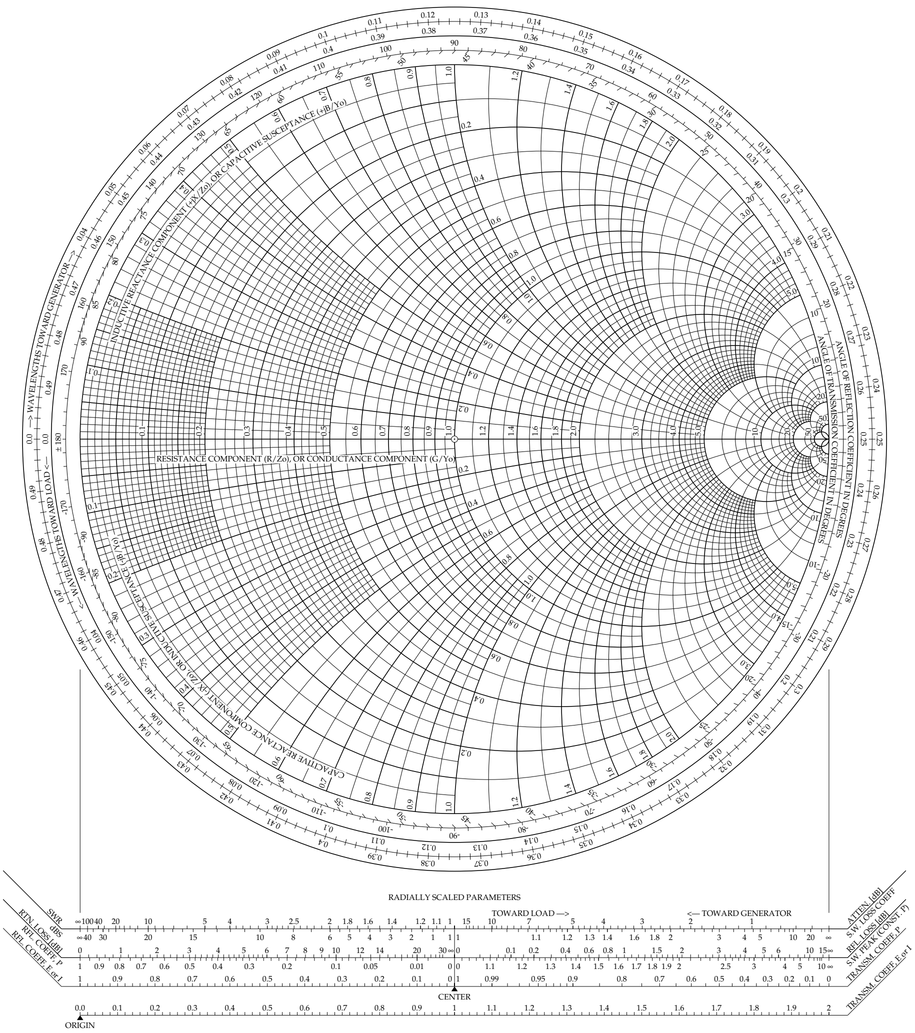

(Source: Wikipedia)

Ex 9-14:

The Complete Smith Chart

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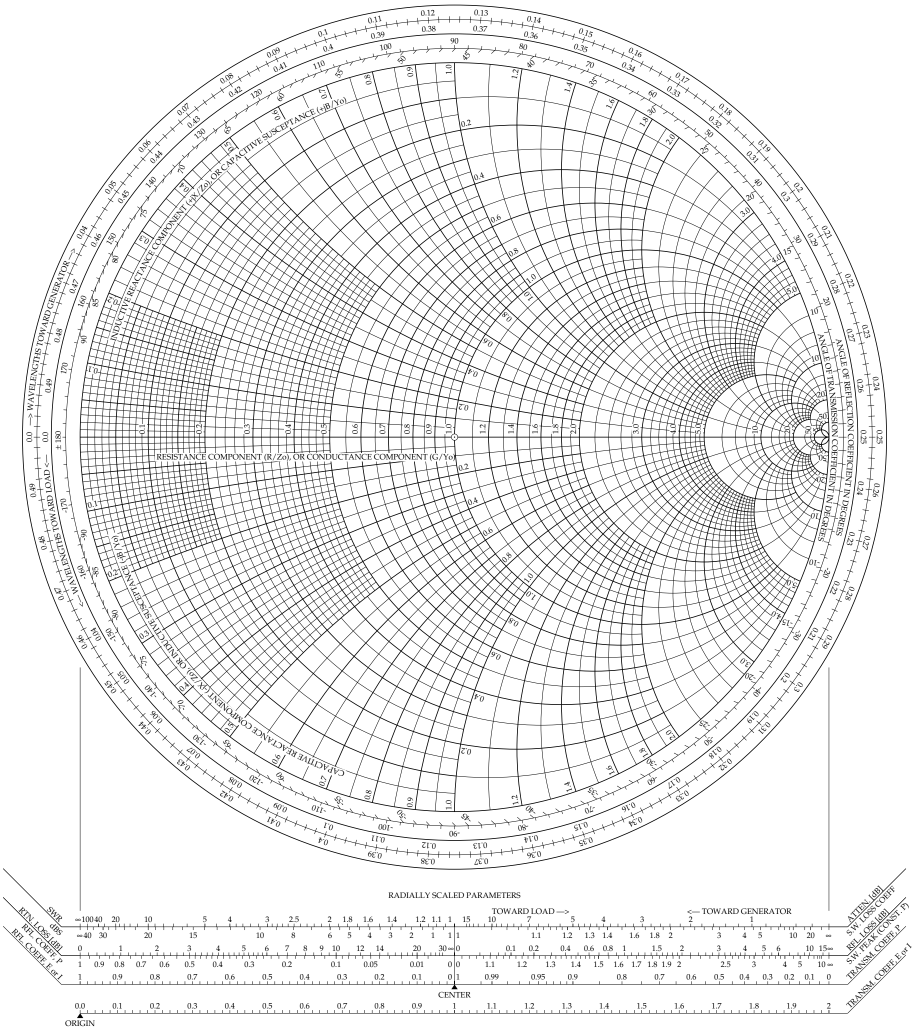
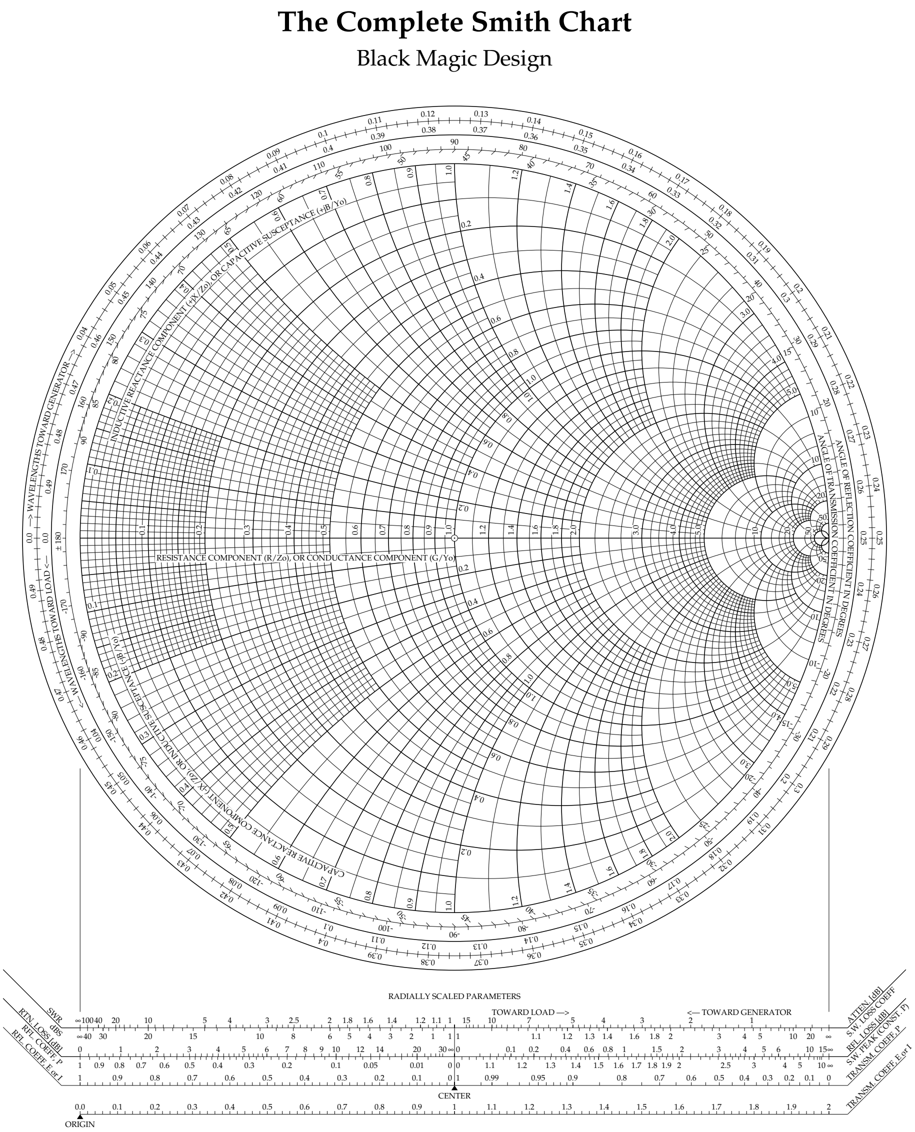


(Source: Wikipedia)

Ex 9-15: Solving Ex 9-9 using Smith Chart



(Source: Wikipedia)

6. Normalized admittance (p. 499)

   Smith Chart relates Γ and normalized load impedance $z_L = Z_L/Z_0$. Now consider the load impedance Y_L. The normalized load impedance can be defined as

   On Smith Chart,

   - z_L and y_L are opposite to each other on the $|\Gamma|$-circle.
   - Same Smith Chart can be used to relate Γ and z_L, or Γ' and y_L.
   - $z_L = r + jx \Leftrightarrow y_L = g + jb$

Ex 9-18: $Z_L = 95 + j20\ \Omega$. Find Y_L.

Ex 9-19:

The Complete Smith Chart

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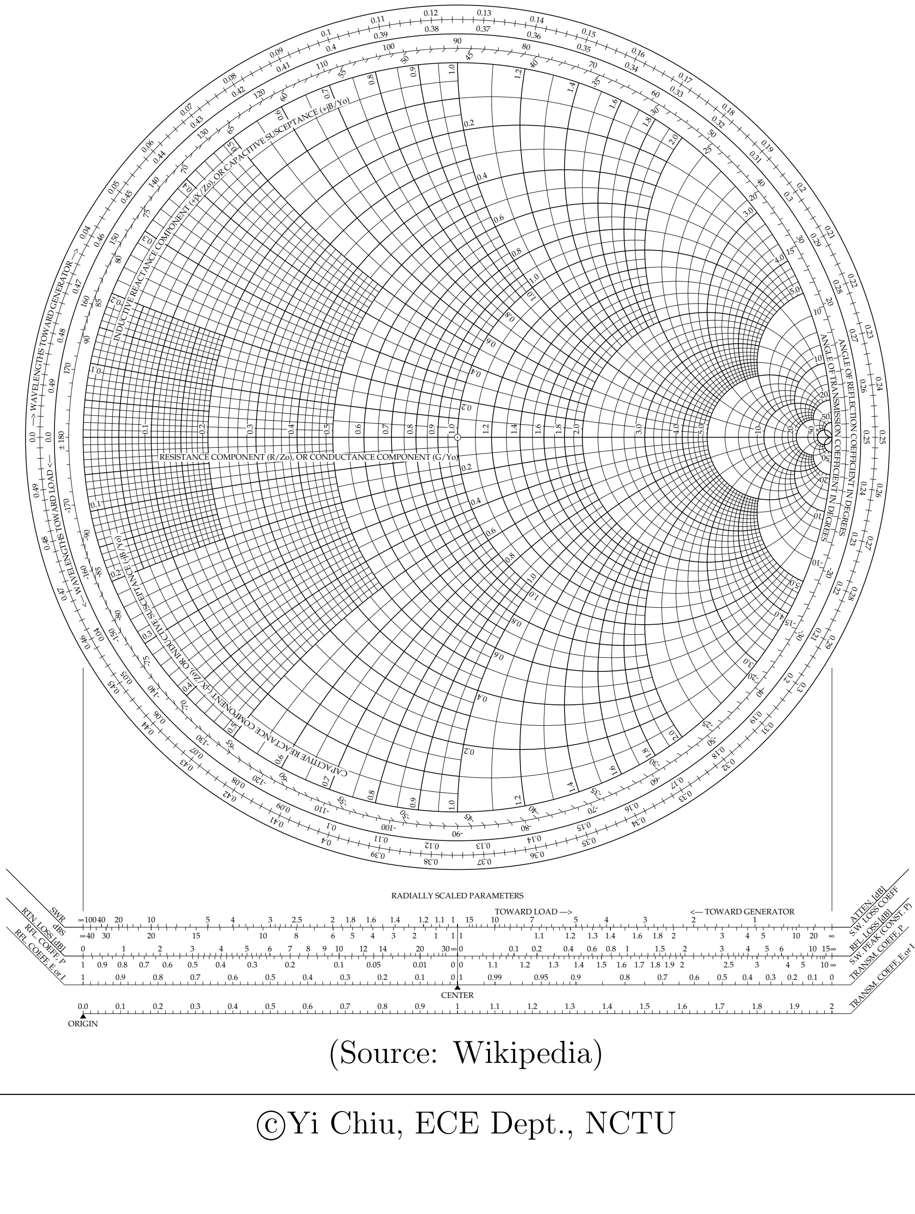


(Source: Wikipedia)

9.7 Transmission Line Impedance Matching

1. reduce reflection and loss
2. improve power transmission efficiency
3. reduce signal interference
4. protect signal generator

9.7.1 Quarter wave transformer

- Frequency sensitive (quarter "wavelength")
- If $R_0, R_0' =$ real, $\Rightarrow R_L$ must also be real $\Rightarrow$ can not be used to match complex load impedance Z_L

Ex 9-17:

(a) For equal power to R_1 and R_2,

(b) Standing wave ratio:

9.7.2 Single stub matching (Stub tuner)

Find ℓ and d such that

In terms of normalized impedance,

But $y_s = g + jb = jb$,

$\Rightarrow$

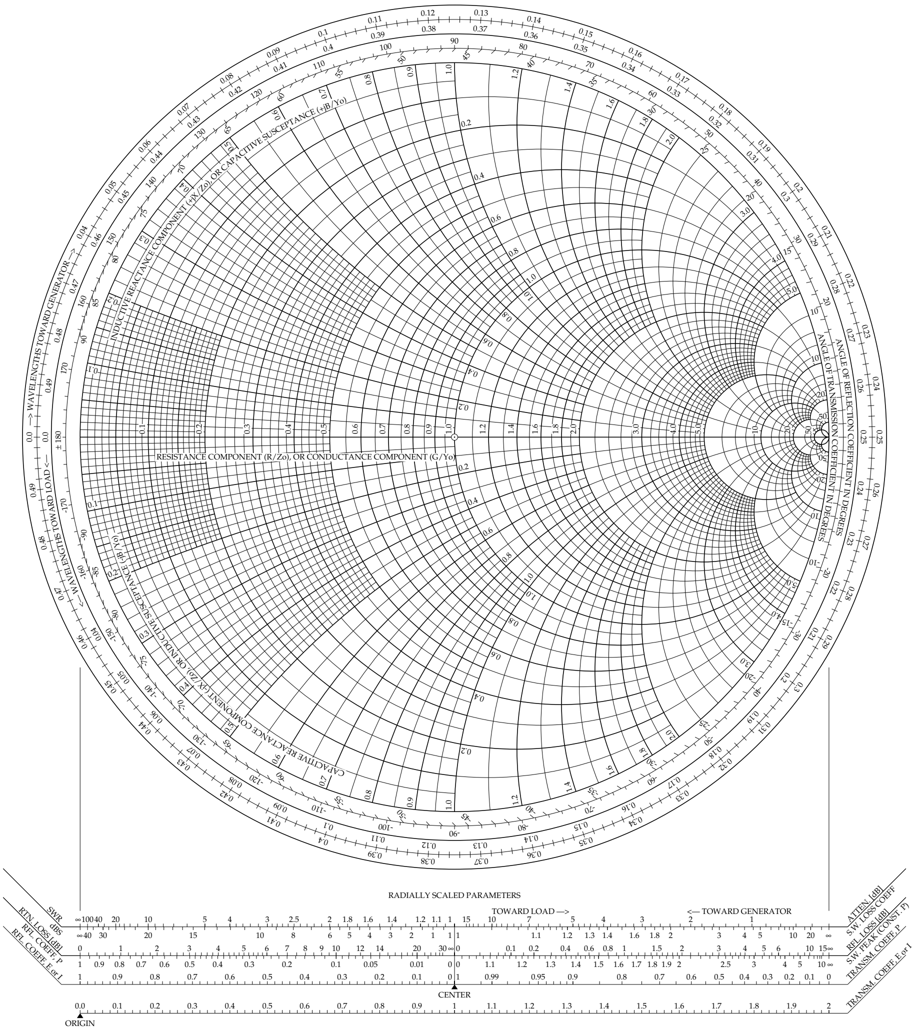
$\Rightarrow$ Find d such that

and ℓ such that

Ex 9-20:

The Complete Smith Chart

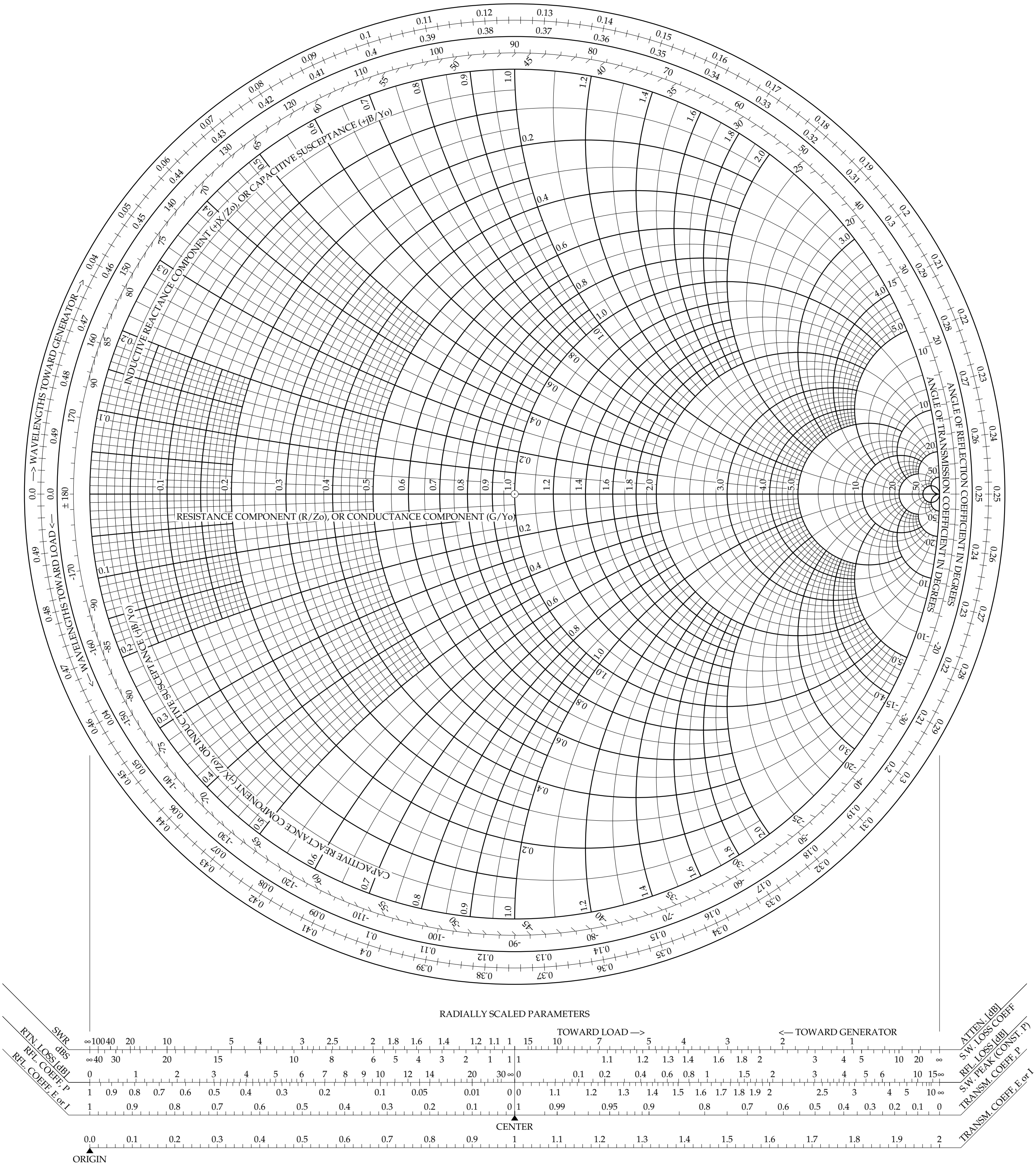
Black Magic Design



(Source: Wikipedia)

The Complete Smith Chart

Black Magic Design



(Source: Wikipedia)