8 Plane Electromagnetic Waves

8.2 Plane Waves in Lossless Media

1. Homogeneous wave equation in free space

   $$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t^2} = 0$$

   For a time harmonic (sinusoidal) field,

   $$\nabla^2\mathbf{E} + k_0^2\mathbf{E} = 0, (\mathbf{E} = \text{phasor})$$

   where $k_0 = \omega\sqrt{\mu_0\epsilon_0} = \omega/c =$ free space wave number.

   In Cartesian corrdinate,

   $$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k_0^2)E_{x,y,z} = 0$$

2. wavefront: set of points/positions in space with same phase

   plane wave: wave with planar wavefront perpendicular to the direction of propagation

   spherical wave: wave with spherical wavefront

   uniform wave: wave with constant **E** and **H** field magnitude across the wavefront

3. Now consider a uniform plane wave with constant E_x on the wavefront xy plane,

   → General solution in phasor form is

   where E_0^+ and E_0^- are complex constant to be determined from boundary or initial conditions.

4. The real-time solution represented by the first phasor term is

   = wave traveling in the $+z$ direction

wave velocity (phase velocity u_p)

$=$

5. Real-time solution represented by the second phasor term is

$$
\begin{aligned}
E_x^-(z,t) &= \Re[E_0^- e^{j(\omega t + k_0 z)}] \\
&= E_0^- \cos(\omega t + k_0 z) \\
&= \text{wave traveling in } -z \text{ direction with phase velocity } u_p = -c
\end{aligned}
$$

- Sign of u_p (or k_0) represents direction of wave propagation.
- If there is only one wave in the $+z$ direction, then $E_0^- = 0$.

6. **H** field can be found from **E** and Maxwell's equations:

$\Rightarrow$

For $+z$ wave,

Similarly for $-z$ wave,

$$H_y^-(z) = -\frac{1}{\eta_0} E_x^-(z)$$

7. $\eta_0 = \dfrac{E_x^+(z)}{H_y^+(z)}$ = intrinsic impedance of the free space

   If η_0 = real $\rightarrow E_x^+(z)$ and $H_y^+(z)$ are in phase

8. Real time solution of the **H** field

   (directions of wave propagation ($\mathbf{a}_z$), electric field ($\mathbf{a}_x$) and magnetic field ($\mathbf{a}_y$) are mutually perpendicular.)

Ex 8-1: A uniform plane wave $\mathbf{E} = \mathbf{a}_x E_x$ propagates in $+z$ direction,

$\epsilon_r = 4, \mu_r = 1, \sigma = 0, f = 100\text{MHz}$

(a) instantaneous expression of **E**

(b) instantaneous expression of **H**

(c) positions for postitive peaks E_x at $t = 10^{-8}$ sec

- $\mathbf{E}$ and $\mathbf{H}$ are perpendicular to each other and both are transverse (perpendicular) to propagation direction
  $\rightarrow$ transverse electromagnetic wave (TEM wave)

9. Doppler effect (see the textbook)

10. For a uniform plane wave propagating in an arbitrary direction in a lossless medium, the solution to the wave equation $\nabla^2\mathbf{E} + k^2\mathbf{E} = 0$ has the general form (from separation of variables, let $\mathbf{E} = \mathbf{E}_0 X(x)Y(y)Z(z)$)

$$\begin{aligned}\mathbf{E}(x, y, z) &= \mathbf{E}_0 e^{-jk_x x} e^{-jk_y y} e^{-jk_z z} \\ &= \mathbf{E}_0 e^{-j(k_x x + k_y y + k_z z)} \\ &= \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{R}}\end{aligned}$$

where $\mathbf{R} = \mathbf{a}_x x + \mathbf{a}_y y + \mathbf{a}_z z =$ position vector

$\mathbf{k} = \mathbf{a}_x k_x + \mathbf{a}_y k_y + \mathbf{a}_z k_z =$ wave vector

$|\mathbf{k}|^2 = k_x^2 + k_y^2 + k_z^2 = k^2 = \omega^2\mu\epsilon$

Wavefront of the wave $\mathbf{E}(x, y, z)$ = set of points with same phase

$\Rightarrow \mathbf{k} = \mathbf{a}_n k$ = wave vector

$\mathbf{a}_n$ = propagation direction

$k = |\mathbf{k}| = \omega\sqrt{\mu\epsilon} = 2\pi/\lambda$ = wave number

11. For a uniform plane wave in a charge free region,

$\Rightarrow \mathbf{E}_0$ is transverse to the propagation direction

12. $\mathbf{H}(\mathbf{R}) =$

where $\eta = \omega\mu/k = \sqrt{\mu/\epsilon}$ (Ω) = intrinsic impedance of the medium,

$\rightarrow \mathbf{H}(\mathbf{R}) = \frac{1}{\eta}\left(\mathbf{a}_n \times \mathbf{E}_0\right) e^{-j\mathbf{k}\cdot\mathbf{R}}$

$\rightarrow \mathbf{H} \perp \mathbf{a}_n$ and $\mathbf{E}$, $\mathbf{a}_n$ is in the direction of $\mathbf{E} \times \mathbf{H}$ (right hand rule)

$\rightarrow$ TEM wave in the $\mathbf{a}_n$ (or $\mathbf{k}$) direction

Ex 8-2: Find $\mathbf{E}(\mathbf{R})$ in terms of $\mathbf{H}(\mathbf{R})$

(自己看)

$\Rightarrow \mathbf{E}(\mathbf{R}) = -\eta\mathbf{a}_n \times \mathbf{H}(\mathbf{R})$

13. Polarization of plane waves (偏極化, 偏振)

    Polarization is the time varying behavior of the orientation of the **E** field at a given point in space. For example, in Ex 8-1,

    The "tip" of **E** field oscillates only along the x axis

    $\rightarrow$ linear polarization (linaerly polarized)

14. Now consider another wave in the $+z$ direction,

    = superposition of two orthogonal linearly polarized wave with y polarization lagging x polarization by $\pi/2$.

    The instantaneous **E** field is

    At a given position $z = 0$,

    Locus of "tip" of $\mathbf{E}(0, t)$ is

    $$\left(\frac{E_1(0,t)}{E_{10}}\right)^2 + \left(\frac{E_2(0,t)}{E_{20}}\right)^2 = \cos^2 \omega t + \sin^2 \omega t = 1$$

    $\rightarrow$ ellipse in the counterclockwise sense

    $\rightarrow$ elliptically polarized if $E_{10} \neq E_{20}$
    circularly polarized if $E_{10} = E_{20}$

    $\rightarrow$ right-hand or positive circularly polarized wave

15. If $E_2(z)$ leads $E_1(z)$ by 90° ($\pi/2$),

    $\mathbf{E}(z) = \mathbf{a}_x E_{10} e^{-jkz} + \mathbf{a}_y j E_{20} e^{-jkz}$

    $\mathbf{E}(0,t) = \mathbf{a}_x E_{10} \cos\omega t - \mathbf{a}_y j E_{20} \sin\omega t$

    - elliptically or circularly polarized wave
    - $\alpha = \tan^{-1} \dfrac{E_2(0,t)}{E_1(0,t)} = -\omega t$ for $E_{20} = E_{10}$

      → left-hand or negative circularly polarized wave

16. If $E_1(z)$ and $E_2(z)$ are in time phase,

    → linearly polarized

17. In general,

    → elliptically polarized

Ex 8-3: Linear and circular waves

Consider a linearly polarized wave propagating in $+z$ direction,

= superposition of a right-hand and a left-hand circular waves

Note:
- AM wave: linearly polarized with **E** field perpendicular to ground
- TV wave: linearly polarized in the horizontal direction
- FM wave: circularly polarized

→ orientation of receiving antenna should be adjusted accordingly

8.3 Plane Waves in Lossy Media

1. wave equation in lossy media

   - $k_c = \omega\sqrt{\mu\epsilon_c}$ = complex wave number (assume μ = real)
   - plane wave e^{jkz} in lossless medium becomes $e^{jk_c z}$ in lossy medium

   We can define a propagation constant γ

   * For lossless media, $\sigma = 0, \epsilon'' = 0 \Rightarrow \alpha = 0, \beta = k = \omega\sqrt{\mu\epsilon}$

   The Helmholtz wave equation becomes

   For a linearly polarized uniform plane wave in $+z$ direction,

   $e^{-\alpha z}$: attenuation factor

   α: attenuation constant (Np/m, 1/m)
   (wave amplitude decresed by a factor of e^{-1} after traveling a distance of $1/\alpha$ meters)

   $e^{-j\beta z}$: phase factor

   β: phase constant (rad/m)
   (similar to the wave number in lossless media)

8.3.1 Low-Loss Dielectrics

1. propagation constant

For a good but imperfect insulator, $\epsilon'' \ll \epsilon'$, or $\sigma/\omega\epsilon \ll 1$,

$\Rightarrow \gamma = \alpha + j\beta = j\omega\sqrt{\mu\epsilon_c} = j\omega\sqrt{\mu\epsilon'}(1 - j\epsilon''/\epsilon')^{-1/2}$

$\approx$

$\Rightarrow$ attenuation constant $\alpha \simeq \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}} \quad \propto \omega$

phase constant $\beta \simeq \omega\sqrt{\mu\epsilon'}\left[1 + \frac{1}{8}\left(\frac{\epsilon''}{\epsilon'}\right)^2\right]$

2. intrinsic impedance

$\eta_c = \sqrt{\frac{\mu}{\epsilon'}}\left(1 - j\frac{\epsilon''}{\epsilon'}\right)^{-1/2}$

$\approx$

$\Rightarrow E_x$ and H_y are not in time phase

3. phase veolcity

$u_p = \frac{\omega}{\beta} \approx \frac{1}{\sqrt{\mu\epsilon'}}\left(1 - \frac{1}{8}\left(\frac{\epsilon''}{\epsilon'}\right)^2\right)$

8.3.2 Good Conductor

1. propagation constant

$\gamma = \alpha + j\beta = j\omega\sqrt{\mu\epsilon}\left(1 + \frac{\sigma}{j\omega\epsilon}\right)^{1/2}$

$\approx$

2. intrinsic impedance

3. phase velocity

$u_p = \frac{\omega}{\beta} \approx \sqrt{\frac{2\omega}{\mu\sigma}}$

4. example: copper

$\Rightarrow$ after a distance of $\delta = 1/\alpha = 0.038$ mm, the wave amplitude will be attenuated by a factor of $e^{-1} = 0.368$.

(At $f = 10$ GHz, $\delta = 0.66\mu$m $\Rightarrow$ high frequency EM wave is attenuated very rapidly in a good conductor.)

5. skin depth or depth of penetration

$\delta =$ distance over which the wave amplitude is attenuated by a factor of $1/e$

$=$

$=$

$=$

$\rightarrow$ $\mathbf{E}, \mathbf{J}$ are distributed near the surface of a conductor at high frequency.

(See Table 8-1)

Ex 8-4: Seawater (Summary)

At $f = 5$ MHz,

$\sigma/\omega\epsilon = 200 \gg 1 \Rightarrow$ good conductor,

attenuation constant $\alpha =$ phase constant $\beta = \sqrt{\pi f \mu \sigma} = 8.89$ (Np/m, rad/m),

wavelength in water $\lambda = 2\pi/\beta = 0.707$ m,

skin depth $\delta = 0.112$ m

(a) At 5 MHz, the wavelength in air is 60 m. It is very difficult to communicate with a submarine due to the small penetration depth in seawater and long wavelength in air (difficult to build efficient tranmission antenna).

(b) After the real-time electric field $E_x(z,t)$ is calculated, it is incorrect to calculate the real-time magnetic field $H_y(z,t)$ from $H_y(z,t) = E_x(z,t)/\eta_c$. Since the complex impedance is defined in the phasor form (frequency domain), the correct formula are

$$H_y(z) = \frac{E_x(z)}{\eta_c},$$

$$H_y(z,t) = \Re\left[\frac{E_x(z)}{\eta_c}e^{j\omega t}\right]$$

8.3.3 Ionized Gas (plasma)

Plasma - assembly of equal numbers of positive ions and negative electrons and possibly other neutral species

- electrons are much lighter than ions
- motion of ions can be neglected and electrons can be viewed as a free electron gas (collision is neglected)
- neutral species, if any, are not affected by electric field

1. From Newton's force law, force on an electron in a time harmonic **E** field is

   In phasor form,

   Displacement **x** from the background positive ions gives rise to a dipole moment

   If there are N electrons per unit volume, the polarization vector is

   Equivalent permittivity of a plasma is

   - propagation constant in a plasma

   - intrinsic impedance

(a) $f < f_p\,, \gamma =$ pure real

$\Rightarrow$ pure attenuation of wave in plasma

$\eta_p = E/H =$ pure imaginary

$\Rightarrow$ plasma is a reactive load

$\Rightarrow$ no power transmission (electrons can move fast enough to screen the EM fields)

($f_p \approx 9\sqrt{N} \approx 0.9 - 9$ MHz for ionosphere)

(b) $f > f_p\,, \gamma =$ pure imaginary

$\Rightarrow$ no attenuation in plasma

Ex 8-5: Communication with space ship

8.4 Group Velocity

1. Phase velocity is defined as $u_p = \omega/\beta$. In a lossless medium, $\beta = \omega\sqrt{\mu\epsilon}$ is a linear function of ω, so the phase velocity is a constant. In cases where the phase constant β is not a linear function of ω (e.g. lossy dielectric, transmissioin line, or waveguide), the phase velocity is not a constant. Therefore, different frequency components in a wave propagate with different phase velocity. This phenomenum is called **dispersion** and the medium or structure is **dispersive**.

2. In information transmission, informatin signals are usually composed of a high frequency carrier surrounded by a signal sideband with finite bandwidth. Therefore, the information waveform with a group of frequencies will be affected by dispersion.

3. Now consider a signal with two frequency components with equal amplitude: $\omega = \omega_0 \pm \Delta\omega, \beta = \beta_0 \pm \Delta\beta$.

(a) phase velocity of carrier, u_p

(b) phase velocity of envelop, u_g

u_g = group velocity of the information signal

4. $\omega - \beta$ diagram of a plasma

Ex 8-6: Narrow band signal in lossy medium

8.5 Flow of Electromagnetic Power and the Poynting Vector

1. From the two curl equaitons,

   it can be shown that for a simple, time invariant medium,

   For a volume V bounded by a surface S,

   $$\Rightarrow \mathbf{P} \triangleq \mathbf{E} \times \mathbf{H} \;= \text{Poynting vector} \\ = \text{power flow per unit area} \\ = \text{power density vector}$$

2. Poynting theorem

   $$-\oint \mathbf{P} \cdot d\mathbf{s} = \frac{\partial}{\partial t} \int_V (w_e + w_m) dv + \int_V P_\sigma dv$$

   where $w_e = \frac{1}{2}\epsilon E^2$ = electric energy density,
   $w_m = \frac{1}{2}\mu H^2$ = magnetic energy density,
   $P_\sigma = \sigma E^2$ = Ohmic power density.

note: (1) $\mathbf{P}$ is perpendicular to $\mathbf{E}$ and $\mathbf{H}$ ($\mathbf{P}$ is parallel to $\mathbf{k}$ for a uniform plane wave in a simple medium)

(2) for lossless media, $P_\sigma = 0$, and energy is stored in the electric and magnetic fields

(3) in a static case, $\partial/\partial t = 0$, all power flow into a closed region is disscipated by the 'equivalent' conduction current (Ohmic loss)

Ex 8-7: Verify the Poynting vector for a straight wire

Consider a section of wire with length ℓ,

8.5.1 Instantaneous and average power density

1. phasor form of field and power density

   $\Rightarrow$ **P** can not be defined as **E** $\times$ **H** in the phasor form.

2. Average power density

   Note that $\Re[\mathbf{A}] \times \Re[\mathbf{B}] = \frac{1}{2}\Re[\mathbf{A} \times \mathbf{B}^* + \mathbf{A} \times \mathbf{B}]$,

   $\Rightarrow$

Consider a time-averaged power density $\mathbf{P}_{\text{av}}$,

3. For a $x-$polarized TEM wave in the $+z$ direction, the Poynting vector expressed in terms of intrinsic impedance of the medium is

8.6 Normal Incidence at a Plane Conducting Boundary

1. Incident wave in medium 1

   Reflected wave in medium 1

   Total **E** field in medium 1

   Boundary conditions at $z = 0$: tangential component of **E** is continuous

   Magnetic field

   $$\Rightarrow \mathbf{H}_1(z) = \mathbf{H}_i(z) + \mathbf{H}_r(z) = \mathbf{a}_y 2\frac{E_{i0}}{\eta_1}\cos\beta_1 z$$

2. Average power

   (a) incident wave:

   reflected wave:

   total average power flow in medium 1

(b) Or, $\eta_1 =$, $\theta_{\eta 1} = -\frac{\pi}{2} \Rightarrow (\mathbf{P}_{\text{av}})_1 = 0$

3. Instantaneous expression of fields

standing wave:

propagation wave:

4. For a standing wave,

$$\begin{cases} \text{zero of } \mathbf{E}_1(z,t): \\ \text{max of } \mathbf{H}_1(z,t): \end{cases}$$

$$\begin{cases} \text{max of } \mathbf{E}_1(z,t): \\ \text{zero of } \mathbf{H}_1(z,t): \end{cases}$$

(a) $\mathbf{E}_1 = 0$ on the surface

(b) $\mathbf{H}_1 = \text{max}$ on the surface

(c) temporal phase difference between $\mathbf{E}$ and $\mathbf{H} = 90°$

(d) spatial separation of $\mathbf{E}$ and $\mathbf{H}$ field patterns $= \lambda/4$

Ex 8-9:

8.7 Oblique Incidence at a Plane Conducting Boundary

1. plane of incidence = plane that contains $\mathbf{k}_i$ and $\mathbf{a}_n$
2. With respect to the plane of incidence, any polarization of $\mathbf{E}_i$ can be decomposed into two components: $\mathbf{E}_{i\perp}$ and $\mathbf{E}_{i\parallel}$

8.7.1 Perpendicular polarizatioin ($\mathbf{E}_{i\perp}$, TE wave)

1. Incident wave

$$\begin{aligned}\mathbf{H}_i(x,z) &= \frac{1}{\eta_1}\mathbf{a}_{ni}\times\mathbf{E}_i(x,z)\\ &= \frac{E_{i0}}{\eta_1}(-\mathbf{a}_x\cos\theta_i+\mathbf{a}_z\sin\theta_i)e^{-j\beta_1(x\sin\theta_i+z\cos\theta_i)}\end{aligned}$$

2. Reflected wave (θ_r = angle of reflection)

$\Rightarrow \mathbf{E}_r(x,z) = \qquad = \mathbf{a}_y E_{r0}e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$

E_{r0} and θ_r can be found from boundary conditions:

$\Rightarrow$

$\Rightarrow$

$\Rightarrow$

$$\begin{aligned}\mathbf{H}_r(x,z) &= \frac{1}{\eta_1}\mathbf{a}_{nr}\times\mathbf{E}_r(x,z)\\ &= \frac{E_{i0}}{\eta_1}(-\mathbf{a}_x\cos\theta_i-\mathbf{a}_z\sin\theta_i)e^{-j\beta_1(x\sin\theta_i+z\cos\theta_i)}\end{aligned}$$

3. Total field

$$\begin{aligned}\mathbf{H}_1(x,z) &= \mathbf{H}_i(x,z)+\mathbf{H}_r(x,z)\\ &= -2\frac{E_{i0}}{\eta_1}\left[\mathbf{a}_x\cos\theta_i\cos(\beta_1 z\cos\theta_i)e^{-j\beta_1 x\sin\theta_i}\right.\\ &\quad\left.+\,\mathbf{a}_z j\sin\theta_i\sin(\beta_1 z\cos\theta_i)e^{-j\beta_1 x\sin\theta_i}\right]\end{aligned}$$

Compared to a plane wave

(a) In the normal direction (z), **E** and **H** form standing wave patterns described by $\beta_{1z}=\beta_1\cos\theta_i$. No average power is transmitted.

(b) In the transverse direction (x), **E** and **H** form propagation wave described by $\beta_{1x}=\beta_1\sin\theta_i$, and

(c) The plane wave in the $x-$direction is non-uniform (interference pattern).

(d) Zeros of $\mathbf{E}_1$:

→ A conducting plane can be inserted at $z = -\frac{m\lambda_1}{2\cos\theta_i}$ without changing the field distribution

→ TE wave in a parallel waveguide

Ex 8-10: (a) Current on the surface of the conductor

→ There is a discontinuity of $\mathbf{H}$ across the interface

→ From B.C., the discontinuity is caused by a surface current $\mathbf{J}_s$

Instantaneous expression of surface current $\mathbf{J}_s$

(b) Poynting vector in medium 1

– E_{1y} and H_{1x} are in time quadrature → no net power in z direction

– $(\mathbf{P}_{\text{av}})_1$ is a function of z because total wave in medium 1 is a non-uniform plane wave

8.7.2 Parallel polarization ($\mathbf{E}_{i\parallel}$, TM wave)

1. Incident wave

2. Reflected wave

$$\begin{aligned}\mathbf{E}_r(x,z) &= E_{r0}(\mathbf{a}_x \cos\theta_r + \mathbf{a}_z \sin\theta_r)e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \\ \mathbf{H}_r(x,z) &= -\mathbf{a}_y \frac{E_{r0}}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}\end{aligned}$$

   B.C.: at $z = 0$,

3. Total wave

   (a) In the normal direction (z), E_{1x} and H_{1y} have standing wave patterns described by $\beta_{1z} = \beta_1 \cos\theta_i$.

   (b) In the transverse direction (x), E_{1z} and H_{1y} are non-uniform propagating wave described by $\beta_{1x} = \beta_1 \sin\theta_i, u_{1x} = \omega/\beta_{1x} = u_1/\sin\theta_i, \lambda_{1x} = 2\pi/\beta_{1x} = \lambda_1/\sin\theta_i$.

   (c) A conducting plane can be inseted at $z = -\frac{m\lambda_1}{2\cos\theta_i}$ to form a parallel plate waveguide for the TM wave

8.8 Normal Incidence at a Plane Dielectric Boundary

1. Incident wave

2. Reflected wave

3. Transmitted wave

$$\begin{cases} \mathbf{E}_t(z) = \mathbf{a}_x E_{t0} e^{-j\beta_2 z}, z > 0 \\ \mathbf{H}_t(z) = \mathbf{a}_y \dfrac{E_{t0}}{\eta_2} e^{-j\beta_2 z} \end{cases}$$

4. Boundary conditions (tangential components)

5. Reflection and transmissin coefficients

$$\begin{cases} \Gamma = \text{reflection coefficient} = \dfrac{E_{r0}}{E_{i0}} = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \\ \\ \tau = \text{transmission coefficient} = \dfrac{E_{t0}}{E_{i0}} = \dfrac{2\eta_2}{\eta_2 + \eta_1} \end{cases}$$

- if $\eta_1, \eta_2 =$ real, $\rightarrow \Gamma > 0$ or < 0 (in- or 180° out-of-phase)

  $\tau > 0$ (in-phase)
- if $\eta_1, \eta_2 =$ complex (dissipative media)

  $\rightarrow \Gamma, \tau =$ complex,

  $\rightarrow$ phase shift in reflected and transmitted waves
- $1 + \Gamma = \tau$
- if medium 2 is a perfect conductor,

6. In general, total field in medium 1 is

7. Pattern of the standing wave

$$\mathbf{E}_1(z) = \mathbf{a}_x E_{i0} e^{-j\beta 1z}(1 + \Gamma e^{j2\beta_1 z})$$

For lossless media, $\eta_1, \eta_2, \Gamma, \tau =$ real,

(a) $\Gamma > 0$ ($\eta_2 > \eta_1$)

(b) $\Gamma < 0$ ($\eta_2 < \eta_1$)

Standing Wave Ratio (SWR)

8. Magnetic field

$$\begin{aligned}\mathbf{H}_1(z) &= \mathbf{H}_i(z) + \mathbf{H}_r(z) \\ &= \mathbf{a}_y \frac{E_{i0}}{\eta_1}(e^{-j\beta_1 z} - \Gamma e^{j\beta_1 z}) \\ &= \mathbf{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1 z}(1 - \Gamma e^{j2\beta_1 z})\end{aligned}$$

In a lossless medium, $|H_1(z)|$ is max (min) where $|E_1(z)|$ is min (max)

9. Transmitted wave in medium 2

Ex 8-11: Power density in lossless media ($\Gamma = \text{real}, \mathbf{P}_{\text{av}} = \frac{1}{2}\Re\{\mathbf{E} \times \mathbf{H}^*\}$)

(a) medium 1

$(\mathbf{P}_{\text{av}})_1 = \frac{1}{2}\Re\{[\mathbf{a}_x E_{i0} e^{-j\beta_1 z}(1+\Gamma e^{j2\beta_1 z})] \times [\mathbf{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1 z}(1-\Gamma e^{j2\beta_1 z})]^*\}$

(b) medium 2

$(\mathbf{P}_{\text{av}})_2 =$

$\Rightarrow 1 - \Gamma^2 = \frac{\eta_1}{\eta_2}\tau^2$ (can also be verified from Eqs. 8-140 and 8-141)

$\Rightarrow$ incident power - reflected power = transmitted power

8.9 Normal Incidence at Multiple Dielectric Interfaces

1. Dielectric coating can be found in eyeglass, camera lens, optical communication systems, and laser systems to reduce or enhance reflection.

2. Consider the following figure. Multiple reflection occurs at $z = 0$ and $z = d$. These waves can be summarized as forward and backward waves.

   Assume an x-polarized incident wave, the total waves in the three regions can be expressed as

   $$\begin{cases} \\ \mathbf{H}_1 = \mathbf{a}_y \frac{1}{\eta_1}(E_{i0}e^{-j\beta_1 z} - E_{r0}e^{j\beta_1 z}) \end{cases}$$

   $$\begin{cases} \\ \mathbf{H}_2 = \mathbf{a}_y \frac{1}{\eta_2}(E_2^+ e^{-j\beta_2 z} - E_2^- e^{j\beta_2 z}) \end{cases}$$

   $$\begin{cases} \\ \mathbf{H}_3 = \mathbf{a}_y \frac{1}{\eta_3} E_3^+ e^{-j\beta_3 z} \end{cases}$$

   - The four unknowns $E_{r0}, E_2^+, E_2^-,$ and E_3^+ can be solved from the boundary conditions on the two interfaces:

     $$\begin{cases} E_{1t}(0) = E_{2t}(0), H_{1t}(0) = H_{2t}(0) \\ E_{2t}(d) = E_{3t}(d), H_{2t}(d) = H_{3t}(d) \end{cases}$$

   - The algebraic procedure is straightforward. But it lacks physical insight and becomes tedious when the number of interfaces increases.

8.9.1 Wave Impedance of the Total Field

1. Wave impedance of total wave is defined as the ratio of total electric field intensity to the toal magnetic field intensity at a particular position z.

   - For for an unbound medium, $Z(z) = \pm\eta$ for a $\pm z$ wave for all z.
   - For two media separated by a plane boundary,

     The wave impedance of total field in medium 1 at a distance z from the boundary is

     At $z = -\ell$ to the left of the boundary,

     $$Z_1(-\ell) = \frac{E_{1x}(-\ell)}{H_{1y}(-\ell)} = \eta_1 \frac{\eta_2 \cos\beta_1\ell + j\eta_1 \sin\beta_1\ell}{\eta_1 \cos\beta_1\ell + j\eta_2 \sin\beta_1\ell}$$

     (note: $\Gamma = (\eta_2 - \eta_1)/(\eta_2 + \eta_1)$)
   - Total wave impedance is a function of both medium properties (η and β) and distance to the boundary (z and ℓ)

8.9.2 Impedance Transformation with Multiple Dielectrics

1. At $z = 0^+$ in medium 2 and looking into $+z$ direction, the situation is same as that of Eqs. 8-169 $\sim$ 8-171. Therefore η_2, η_1, β_1 and ℓ can be replaced by η_3, η_2, β_2 and d.

   $$\Rightarrow Z_2(0^+) = \eta_2 \frac{\eta_3 \cos\beta_2 d + j\eta_2 \sin\beta_2 d}{\eta_2 \cos\beta_2 d + j\eta_3 \sin\beta_2 d}$$

   (η_3 is transformed to $Z_2(0)$)

2. For the incident wave in medium 1, it sees an equivalent infinite medium with instrinsic impedance $Z_2(0)$ at the boundary $z = 0$. Therefore the effective reflection coefficient at $z = 0$ is

   - In comparison to $\Gamma = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$
   - In medium 1, Γ_0 and E_{r0} can be calculated by the transformed impedance $Z_2(0)$.
   - Fields in medium 2 and medium 3 can be calculated from B.C.

Ex 8-12: Find η_2 and d for anti-reflection coating

$\Rightarrow \Gamma_0 = 0, Z_2(0) = \eta_1$

$\Rightarrow \eta_2(\eta_3 \cos\beta_2 d + j\eta_2 \sin\beta_2 d) = \eta_1(\eta_2 \cos\beta_2 d + j\eta_3 \sin\beta_2 d)$

$\Rightarrow$

(a) $\eta_1 = \eta_3, \eta_2 = \sqrt{\eta_1\eta_3} = \eta_1$

$\Rightarrow$trivial solution

(b) $\eta_1 = \eta_3, \eta_2 \neq \sqrt{\eta_1\eta_3}$

(c) $\eta_1 \neq \eta_3$

- Arbitray η_2 may not be available
- For optical applications, complex multiple coating is necessary for wideband or other types of filter

8.10 Oblique Incidence at a Plane Dielectric Boundary

1. Snell's law (see textbook)

$$\begin{cases} \theta_r = \theta_i \qquad \text{(reflection)} \\ n_1 \sin\theta_i = n_2 \sin\theta_t \qquad \text{(refraction)} \end{cases}$$

- n_1, n_2 = index of refraction and

$$\frac{n_1}{n_2} = \frac{u_{p2}}{u_{p1}} = \frac{\beta_1}{\beta_2} = \sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}}$$
$$= \sqrt{\frac{\epsilon_1}{\epsilon_2}} = \sqrt{\frac{\epsilon_{1r}}{\epsilon_{2r}}} \quad \text{for non-magnetic media}$$

- Both Snell's laws are independent of polarization.

8.10.1 Total reflection

1. If $\epsilon_1 > \epsilon_2$ ($n_1 > n_2$), there is a critial incident angle θ_c such that $\theta_t = \pi/2$,

2. When $\theta_i > \theta_c$, there is no real solution for θ_t
   $\rightarrow$ no transmitted wave in medium 2 from a geometric point of view

3. From a physical point of view, when $\theta_i > \theta_c$,

   In this case, the transmitted wave in medium 2 is

Ex 8-14: dielectric waveguide (e.g. optical fibers)

When $\theta_1 > \theta_c$, power can be guided along the waveguide by total internal reflection. The condition for waveguiding is

8.10.2 Perpendicular Polarization (TE wave)

1. Incident wave

2. Reflected wave

$$\begin{cases} \mathbf{E}_r(x,z) &= \mathbf{a}_y E_{r0} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \\ \mathbf{H}_r(x,z) &= \frac{E_{r0}}{\eta_1}(\mathbf{a}_x\cos\theta_r + \mathbf{a}_z\sin\theta_r)e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \end{cases}$$

3. Transmitted wave

$$\begin{cases} \mathbf{E}_t(x,z) &= \mathbf{a}_y E_{t0} e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)} \\ \mathbf{H}_t(x,z) &= \frac{E_{t0}}{\eta_2}(-\mathbf{a}_x\cos\theta_t + \mathbf{a}_z\sin\theta_t)e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)} \end{cases}$$

4. Boundary conditions: tangential components of $\mathbf{E}, E_y$, and $\mathbf{H}, H_x$, are continuous across the boundary

$$\Rightarrow \begin{cases} E_{i0}e^{-j\beta_1 x\sin\theta_i} + E_{r0}e^{-j\beta_1 x\sin\theta_r} = E_{t0}e^{-j\beta_2 x\sin\theta_t} \\ \frac{1}{\eta_1}(-E_{i0}\cos\theta_i e^{-j\beta_1 x\sin\theta_i} + E_{r0}\cos\theta_r e^{-j\beta_1 x\sin\theta_r}) = -\frac{E_{t0}}{\eta_2}\cos\theta_t e^{-j\beta_2 x\sin\theta_t} \end{cases}$$

"phase matching" condition

$\Rightarrow$

$$\Rightarrow \begin{cases} \frac{E_{r0}}{E_{i0}} &= \Gamma_\perp &= \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} = \frac{\eta_2/\cos\theta_t - \eta_1/\cos\theta_i}{\eta_2/\cos\theta_t + \eta_1/\cos\theta_i} \\ \frac{E_{t0}}{E_{i0}} &= \tau_\perp &= \frac{2\eta_2/\cos\theta_t}{\eta_2/\cos\theta_t + \eta_1/\cos\theta_i} \quad \text{(Fresnel's eq.)} \end{cases}$$

- For normal incidence, $\theta_i = 0 = \theta_r = \theta_t$
- $1 + \Gamma_\perp = \tau_\perp$
- Brewster angle ($\theta_{B\perp}$)

According to Snell's law,

(For non-magnetic materials, $\mu_1 = \mu_2 = \mu_0, \sin^2\theta_{B\perp} \to \infty, \theta_{B\perp}$ does not exist)

8.10.3 Parallel Polarization (TM wave)

1. Similarly,

$$\begin{cases} \theta_r = \theta_i \\ \dfrac{\sin\theta_t}{\sin\theta_i} = \dfrac{\beta_1}{\beta_2} = \dfrac{n_1}{n_2} \end{cases}$$

$$\begin{cases} \Gamma_\parallel = \dfrac{E_{r0}}{E_{i0}} = \dfrac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} \\ \tau_\parallel = \dfrac{E_{t0}}{E_{i0}} = \dfrac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} \end{cases}$$

2. Brewster's angle

- $1 + \Gamma_{\|} = \tau_{\|} \dfrac{\cos\theta_t}{\cos\theta_i}$
- $\Gamma_{\perp} \neq \Gamma_{\|}, \tau_{\perp} \neq \tau_{\|}$ unless $\theta_i = \theta_r = \theta_t = 0$ (normal incidence)
- $|\Gamma_{\perp}|^2 > |\Gamma_{\|}|^2$ except at $\theta_i = 0$,

  $\rightarrow$ unpolarized incident wave upon reflection

  $\rightarrow$ more reflected power in the $\perp$ polarization than in the $\|$ polarization

Ex 8-15: Reflection from water