7 Time-Varying Fields and Maxwell's Eqs.

7.1 Introduction and review

1. Electrostatic and magnetostatic fields

$$\begin{cases} \nabla \times \mathbf{E} = 0 & \text{(electrosttic field } \mathbf{E} \text{ is conservative)} \\ \nabla \cdot \mathbf{D} = \rho & (\rho \text{ is the source of electrostatic field)} \end{cases}$$

$$\begin{cases} \nabla \cdot \mathbf{B} = 0 & \text{(no magnetic monopole)} \\ \nabla \times \mathbf{H} = \mathbf{J} & (\mathbf{J} \text{ is the source of magnetostatic field)} \end{cases}$$

2. Constitutional relations (linear isotropic medium)

$$\begin{cases} \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{H} = \mathbf{B}/\mu \end{cases}$$

3. Continuity relations (conservation of charge)

   $\nabla \cdot \mathbf{J} + \partial\rho/\partial t = 0$

4. Lorentz's force equation

   $\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$

5. Current density

   conduction current: $\mathbf{J} = \sigma \mathbf{E}$

   convection current: $\mathbf{J} = \rho \mathbf{u}$

6. Potential function

$$\begin{cases} \nabla \times \mathbf{E} = 0 \;\rightarrow\; \mathbf{E} = -\nabla V \\ \nabla \cdot \mathbf{B} = 0 \;\rightarrow\; \mathbf{B} = \nabla \times \mathbf{A} \end{cases}$$

* In static cases, electric and magnetic fields are independent physical quantities.

7.2 Faraday's Law of Electromagnetic Induction

1. Electrostatic field, electrostatic potential, and electromotive force (emf) (ref: §5-3)

   (a) Electrostaic field is a conservative field. The total energy gained by an electric charge moving along a closed loop in a lossless medium in an electrostatic field is zero. However, when an electric energy source (such as a battery) and a load (such as a resistor) are connected to form a closed loop, the charge will gain a net energy (which is then lost in the load and leaves the system) from the source when moving around the closed loop. Therefore, the 'equivalent' electric field due to the energy source is not conservative.

   (b) Electromotive force (emf, $\mathcal{V}$) is used to describe the capability of the source to provide energy to the charges. It is defined as the energy gained by a unit charge as it moves across the source.

2. Faraday's experiment: a current was induced in the circuit when the magnetic flux linkage changed.

   For an open surface S with boundary C, the electromotive force (emf) $\mathcal{V}$ induced in C is defined as the energy gained by a unit charge traveling around the contour C once:

   $\mathcal{V} =$

The magnetic flux linkage in S is

⇒ Faraday's law can be expressed as

- one of the fundamental laws (Coulomb's law, Lorentz's law, etc.)
- link of **E** and **B** through time derivative
- negative sign follows the Lenz's law

⇒ Fundamental postulates for electromagnetic induction

integral form:

differential form:

- if $\partial\mathbf{B}/\partial t \neq 0 \Rightarrow \nabla \times \mathbf{E} \neq 0, \mathbf{E}$ is not conservative
- if $\partial\mathbf{B}/\partial t = 0 \Rightarrow$ electrostatics, conservative field

(a) Lenz's law

→ induced **E** and I in the negative direction

→ induced **B** and Φ (by **E** and I) in the negative direction

→ induced current and flux are opposing the change of Φ

(b) eddy current and transformer

A time-varying magnetic field induces an emf and a non-conservative electric field in a conductor such as the ferromagnetic cores in transformers.

→ an eddy current $\mathbf{J} = \sigma\mathbf{E}$ around the magnetic field

→ ohmic loss and local heating for $\sigma \neq \infty$

→ power loss can be reduced with laminated cores

3. Moving circuits in time varying fields (§7-2.4)

   In calculating the magnetic flux Φ in a time varying situation, both field ($\mathbf{B}$) and circuits (S and C) can vary. It can be shown that the induced emf $\mathcal{V} = -d\Phi/dt$ can be written as

$$\begin{aligned} \mathcal{V} &= \oint_C \mathbf{E} \cdot d\boldsymbol{\ell} \\ &= -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} + \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell} \\ &= \text{transformer emf } \mathcal{V}_a \\ &\quad + \text{motional (or flux-cutting) emf } \mathcal{V}'_a \end{aligned}$$

Ex 7-4: Rotating loop in changing magnetic field (see textbook)

7.3 Maxwell's Equations

1. Elecrtromagnetic postulates including Faraday's law

2. Continuity equation (conservation of charge) must be satisfied at all time. Therefore,

   $\Rightarrow$ Postulates must be modified

$\Rightarrow$ Macroscopic electromagnetic phenomena can be fully described by

(a) Maxwell's equations

- Differential form

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t} \end{cases}$$

* $\rho, \mathbf{J}$: free charge and volume current density
* two divergence equations can be derived from two curl equations

* constitutive relations (for linear isotropic medium)

$$\begin{cases} \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{H} = \mathbf{B}/\mu \end{cases}$$

- Integral form

$$\begin{cases} \oint_S \mathbf{D} \cdot d\mathbf{s} = Q & \text{(Gauss's law)} \\ \oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\dfrac{d\Phi}{dt} & \text{(Faradys's law)} \\ \oint_S \mathbf{B} \cdot d\mathbf{s} = 0 & \text{(No isolated magnetic charge)} \\ \oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I + \int_S \dfrac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s} & \text{(Ampere's law)} \end{cases}$$

(b) Continuity equation (conservation of charge)

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$$

(c) Lorentz's force equation

$$\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

- Conduction current density in a conductor: $\mathbf{J} = \sigma \mathbf{E}$
  Convection current density in vacuum: $\mathbf{J} = \rho \mathbf{u}$

- In static cases, electric and magnetic fields are independent physical quantities. In time-varying cases, electric fields and magnetic fields are coupled.

Ex 7-5: Displacement current in a capacitor

(a) conduction current i_c and displacement current i_d

(b) Find **H** around the wire (from Ampere's law)

C is the common boundary of open surfacs S_1 and S_2.

$\Rightarrow \oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = \oint_{S_1} \nabla \times \mathbf{H} \cdot d\mathbf{s} = \oint_{S_2} \nabla \times \mathbf{H} \cdot d\mathbf{s}$

i. calculation of **H** from C and S_1

ii. calculation of **H** from C and S_2

$\Rightarrow$

(If displacement current $\mathbf{J}_D$ were not included in the Maxwell's equations, **H** field calculated from S_1 and S_2 would be different.)

7.5 Electromagnetic Boundary Conditions

$$\begin{cases} \oint_S \mathbf{D} \cdot d\mathbf{s} = Q \\ \oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{s} \\ \oint_S \mathbf{B} \cdot d\mathbf{s} = 0 \\ \oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = \int_S (\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}) \cdot d\mathbf{s} \end{cases}$$

1. B.C. can be found by applying the integral forms of Maxwell's Eqs. to a small loop or box across the boundary of two media. The results are (same as B.C. in static fields):

   - Tangential components

   $$\begin{cases} E_{1t} = E_{2t} & \text{(a)} \\ \mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s & \text{(b)} \end{cases}$$

   - Normal components

   $$\begin{cases} \mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s & \text{(c)} \\ B_{1n} = B_{2n} & \text{(d)} \end{cases}$$

   - In time varying cases, (a) is equivalent to (d), (b) is equivalent to (c).

2. Interface between two lossless linear media ($\sigma = 0, \rho_s = 0, \mathbf{J}_s = 0$)

$$\begin{cases} E_{1t} = E_{2t},\ H_{1t} = H_{2t} \\ D_{1n} = D_{2n},\ B_{1n} = B_{2n} \end{cases}$$

3. Interface between dielectric and perfect conductor

Metal

$$\begin{cases} E_{2t} = 0 \\ H_{2t} = 0 \\ D_{2n} = 0 \\ B_{2n} = 0 \end{cases}$$

Dielectric

$$\begin{cases} E_{1t} = 0 \\ \mathbf{a}_{n2} \times \mathbf{H}_1 = \mathbf{J}_s, |\mathbf{H}_1| = |\mathbf{H}_{1t}| = |\mathbf{J}_s| \\ \mathbf{a}_{n2} \cdot \mathbf{D}_1 = \rho_s, |\mathbf{E}_1| = |\mathbf{E}_{1n}| = \rho_s/\epsilon_1 \\ B_{1n} = 0 \end{cases}$$

(time varying component only)

7.6 Wave Equations and Solutions

1. Source-free wave equations

   In source free regions ($\rho = 0, \mathbf{J} = 0$), the Maxwell's equations are reduced to

   $\Rightarrow$

   $\Rightarrow$

   Similarly, $\nabla^2 \mathbf{H} - \frac{1}{u^2}\frac{\partial^2 \mathbf{H}}{\partial t^2} = 0$

   - homogeneous wave equations for **E** and **H**
   - wave propagation speed $= 1/\sqrt{\mu_0 \epsilon_0} = \text{C}$ in vacuum

2. In Cartician coordinate,

$$\left.\begin{aligned} \nabla^2 H_x - \frac{1}{u^2}\frac{\partial^2 H_x}{\partial t^2} = 0 \\ \nabla^2 H_y - \frac{1}{u^2}\frac{\partial^2 H_y}{\partial t^2} = 0 \\ \nabla^2 H_z - \frac{1}{u^2}\frac{\partial^2 H_z}{\partial t^2} = 0 \end{aligned}\right\} \text{wave equation: } \nabla^2 f - \frac{1}{u^2}\frac{\partial^2 f}{\partial t^2} = 0$$

3. Wave equation in one dimension: $\frac{\partial^2 f}{\partial x^2} - \frac{1}{u^2}\frac{\partial^2 f}{\partial t^2} = 0$

   The solution of the wave equation (wave function) has the general form $f(x,t) = f(kx \pm \omega t)$, where $k^2 - \omega^2/u^2 = 0$. For a sinusoidal solution,

$f(x,t) = f_0 \sin(kx - \omega t)$, where

4. Phase velocity

   $f(x,t) = f_0 \sin(kx - \omega t) = \sin(\phi)$, where ϕ = phase.

   At fixed t,

   If we look at the velocity of the constant-phase point P,

5. Propagation and time delay

   Consider two points x_1 and x_2

   $\Rightarrow$ the wave function value at x_2 is a delayed version of that at x_1

6. Waves in 2D and 3D

7.7 Time-Harmonic (Sinusoidal) Fields

1. In circuit analysis, a sinusoidal time-varying (time harmonic) signal can be written as

   where $I_s = I_0 e^{j\phi}$ (= phasor) contains both amplitude (I_0) and phase (ϕ) information. The phase ϕ represents time delay with respect to a reference starting time instant.

2. In phasor analysis, I_s is used to represent the signal. All time derivative operation d/dt can be replaced by a multiplication operation by $j\omega$. The real signal in time domain can be obtained by taking the real part of the product of I_s and $e^{j\omega t}$, i.e. $i(t) = \Re[I_s e^{j\omega t}] = \Re[I_0 e^{j(\omega t+\phi)}]$.

3. Similarly, for a time harmonic EM field, the field at each point $\mathbf{R}$ can be written as

   where

   - $\mathbf{E}(\mathbf{R})$ is the vector phasor that contains amplitude, direction, and phase information of the field at $\mathbf{R}$
   - Real solution of the fields can be obtained by taking the real part of $\mathbf{E}(\mathbf{R})e^{j\omega t}$, i.e. $\mathbf{E}(\mathbf{R}, t) = \Re[\mathbf{E}(\mathbf{R})e^{j\omega t}] = \Re[\mathbf{E}_0(\mathbf{R})e^{j\omega t+\phi(\mathbf{R})}]$, where $\phi(\mathbf{R})$ is now a position-dependent phase delay.
   - A vector phasor has three scalar phasor components:

     where E_x, E_y, E_z are scalar phasors and

⇒ In terms of phasors, the Maxwell's Eqs. can be written as instantaneous form phasor form

$$\begin{cases} \nabla \times \mathbf{E} = -\mu \dfrac{\partial \mathbf{H}}{\partial t} \\ \nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t} \\ \nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon} \\ \nabla \cdot \mathbf{H} = 0 \end{cases}$$

4. In time harmonic fields, the source free wave equations can be written as (from Eqs. 7-81, 7-82, $\nabla^2 \mathbf{E} - \mu\epsilon \dfrac{\partial^2 \mathbf{E}}{\partial t^2} = 0$)

   (homogeneous vector Helmholtz's equations)

   where $k = \omega\sqrt{\mu\epsilon} = \omega/u = 2\pi/\lambda =$ wave number = spatial frequency

5. If the medium is conductive, $\mathbf{J} = \sigma \mathbf{E} \neq 0$,

   where

   - If permittivity can be a complex number, the same form of Maxwell's Eqs. can be applied to conductive and non-conductive materials.
   - Imaginary part of ϵ_c represents energy loss (Ohmic loss, frictional damping, ...). An equivalent conductivity for all losses can be defined as $\sigma \triangleq \omega\epsilon''$.
   - Similarly, magnetic permeability can be a complex number $\mu = \mu' - j\mu''$
   - For $\mu' \gg \mu'', \mu \sim \mu'$, complex wave number k_c in a lossy dielectric medium becomes

6. Loss tangent $\tan\delta_c = \epsilon''/\epsilon'$

   loss angle $\delta_c = \tan^{-1}\epsilon''/\epsilon'$

   - If $\epsilon''/\epsilon' = \sigma/\omega\epsilon' \gg 1 \Rightarrow$ good conductor

     If $\epsilon''/\epsilon' = \sigma/\omega\epsilon' \ll 1 \Rightarrow$ good insulator

   - Dielectric properties may depend on frequency

Ex 7-8 (自己看)

7. Electromagnetic spectrum