involve no absurdity and create no confusion of ideas."

He saw in the workings of nature confirmation of his ideas. Were not people black in hot climates and white in cold ones? The black skin of the Negro allowed him to radiate efficiently and thereby keep cool, while white skin was an efficient reflector of frigorific radiation and hence defended the white man from the cold. Rumford was so sure of his conclusions that he carried his convictions to the logical conclusion of always wearing white clothing in cold weather, much to the derisive amusement of the Parisian society in which he moved in later years.

Rumford had no concept of heat as a random motion. He felt that heat was primarily set up by the harmonic vibrations of the "fibers of the metal,"[3] was transmitted through solids and radiated from them in the same manner as acoustic waves. He did not feel that these same waves could be set up in fluids. In fact, he carried out a long series of experiments[8] showing that gases and liquids (including mercury!) were perfect nonconductors and that their only mode of communicating heat was by convection.[9] He felt that what heat was transmitted through fluids at rest was due only to the conduction of thermal vibrations in the all-pervading ether, and he was strengthened in this belief by his showing that heat passed almost as easily through a Torricellian vacuum as through air.[10] By 1800 he was completely convinced that heat was a vibratory motion, analogous in every way to acoustical oscillations.

[8] Count Rumford, Essay VII, Part II, Cadel and Davies, London, 1798.
[9] S. C. Brown, Am. J. Phys. **15**, 273 (1947).
[10] Count Rumford, Trans. Roy. Soc. (London) **76**, 273 (1786).

A Direct Treatment of the Foucault Pendulum

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The equations of motion of the Foucault pendulum are set up in polar coordinates. The oscillation is shown to be simple harmonic for a particular angular velocity, $-\Omega\sin\phi$, where Ω is the angular velocity of rotation of the earth and ϕ is the latitude. In general, the motion involves a constant areal velocity c and a nonlinear oscillation given by $\ddot{r}+\omega^2r-c^2/r^3=0$. This equation is integrated through the energy equation and shown to give the same precession as in the harmonic case.

FOUCAULT'S great pendulum of 1851, swinging under the dome of the Panthéon in Paris, gave a dramatic proof of the rotation of the earth. The slow precession of the plane of oscillation of a freely suspended pendulum is still followed with interest, when the experiment is repeated in observatories and science museums throughout the world, and many elementary physics textbooks discuss the phenomenon as a laboratory demonstration of the earth's rotation. As has been pointed out recently,[1] it is not correct to say simply that the pendulum continues to swing in a fixed plane or that the horizontal path of the bob maintains a fixed direction in space. It is quite possible to give a reasoned geometrical explanation[2] of the observed precession of the plane of oscillation, but this is cumbersome and rarely done.

The more advanced books offer a variety of treatments of the Foucault pendulum. With the traditional method of starting the oscillations, by burning a string which holds the bob out from the equilibrium position, the motion should follow a pointed-star pattern as described in a detailed derivation[3] of the special cases by Kimball in this journal. The author points out that, in practice, his different patterns are indistinguishable and amount to a simple harmonic motion in a slowly rotating plane. A complete

[1] Wylie, Pop. Astron. **57**, 170 (1949).
[2] Grimsehl, *A Textbook of Physics* (Blackie & Son Limited, London and Glasgow, 1932), Vol. 1, p. 165.
[3] W. S. Kimball, Am. J. Phys. **13**, 271 (1945).

development[4] from the differential equations of the spherical pendulum shows the dependence on the initial conditions. This derivation, a concise vector treatment,[5] and a development in complex quantities[6] have in common the introduction of a rotation of the reference axes to compensate for the rotation of the earth. There follows an alternate direct treatment of this historic and important example of motion on a rotating earth, quite within the reach of students of intermediate mechanics.

THE CORIOLIS FORCE

By the theorem of Coriolis (1829), motion relative to moving axes can be treated by the ordinary equations of motion by the addition, to the actual forces, of fictitious forces capable of producing accelerations equal and opposite to the acceleration of moving space and the compound centripetal acceleration. This is, of course, an application of the principle, introduced by d'Alembert in his *Traité de Dynamique* (1743), that the system of external forces is, as a whole, in dynamic equilibrium with the inertial reactions of the accelerated masses. For the dynamics of a particle, d'Alembert's principle is implicit in Newton's second law (1687) and the theorem of Coriolis involves only the kinematics of relative motion. A modern derivation can be found in any standard textbook.

For an observer on the uniformly rotating earth, the first fictitious force reduces to the ordinary centrifugal force of elementary mechanics. It is constant at a given station and is included in the resultant gravitational force, the weight of the body. The second, the Coriolis or deflecting force, depends on the latitude and the velocity of the body; it is familiar to many physics students through its dominant role in the calculation of the geostrophic wind velocity in meteorology. For motion in a horizontal plane, the horizontal component of the Coriolis force is at right angles to the velocity v of the body and has a magnitude $2mv\Omega \sin\phi$ where, for a latitude ϕ, $\Omega \sin\phi$ is the vertical component of the angular velocity of the earth.

[4] Webster, *The Dynamics of Particles and of Rigid, Elastic, and Fluid Bodies* (B. G. Teubner, Leipzig, Germany, 1904), p. 323.

[5] Page, *Introduction to Theoretical Physics* (D. Van Nostrand and Company, Inc., New York, 1935), second edition, p. 107.

[6] Joos, *Theoretical Physics* (Blackie & Son Limited, London and Glasgow, 1951), second edition, p. 814.

THE EQUATIONS OF MOTION

With a long suspension and a small maximum displacement from the equilibrium position, the motion of a pendulum bob is effectively in the horizontal plane. The simple pendulum of mass m, length l, and small amplitude of oscillation is assumed to execute simple harmonic motion under the influence of a restoring force, mgr/l, proportional to the displacement r. As this approximation is always introduced sooner or later, we shall distinguish the Foucault pendulum proper from the spherical pendulum by postulating its validity in what follows.

Using polar coordinates for the position of the pendulum bob in the horizontal plane, we have at any point radial and transverse velocities, $\dot{r}$ and $r\dot{\theta}$, and the corresponding accelerations, $\ddot{r}-r\dot{\theta}^2$ and $r\ddot{\theta}+2\dot{r}\dot{\theta}$. With the inclusion of the Coriolis forces, the equations of motion are

$$m(\ddot{r}-r\dot{\theta}^2) = -mgr/l + 2mr\dot{\theta}\Omega \sin\phi \qquad (1)$$

and

$$m(r\ddot{\theta}+2\dot{r}\dot{\theta}) = -2m\dot{r}\Omega \sin\phi. \qquad (2)$$

THE HARMONIC OSCILLATOR

The equations of motion reduce to

$$\ddot{r}+r(g/l-\dot{\theta}^2-2\dot{\theta}\Omega \sin\phi)=0 \qquad (3)$$

and

$$r\ddot{\theta}+2\dot{r}(\dot{\theta}+\Omega \sin\phi)=0. \qquad (4)$$

Equation (4) is satisfied by a constant angular velocity, $-\Omega \sin\phi$, of the pendulum bob since $\dot{\theta}+\Omega \sin\phi = 0$ gives $\ddot{\theta}=0$. Substituting this value for $\dot{\theta}$ in Eq. (3), we get

$$\ddot{r}+\omega^2 r=0, \qquad (5)$$

where $\omega^2 = g/l+\Omega^2 \sin^2\phi$. This represents a simple harmonic motion along a radius vector with a period T given by

$$T=2\pi(g/l+\Omega^2 \sin^2\phi)^{-\frac{1}{2}}. \qquad (6)$$

For motion from a maximum displacement A at an angle θ_0, the complete solution is given by

$$r=A \cos(g/l+\Omega^2 \sin^2\phi)^{\frac{1}{2}}t \qquad (7)$$

and

$$\theta=\theta_0-\Omega t \sin\phi. \qquad (8)$$

THE NONLINEAR OSCILLATOR

In general, the solution of Eq. (4) is of the form

$$r^2(\dot{\theta}+\Omega \sin\phi)=c, \quad (9)$$

where c is a constant areal velocity. (This is not the rate at which the radius vector sweeps out area, which is simply $\frac{1}{2}r^2\dot{\theta}$.) Substituting from Eq. (9) into Eq. (3), we get the equation of a nonlinear oscillator,

$$\ddot{r}+\omega^2 r-c^2/r^3=0. \quad (10)$$

This is not one of the common nonlinear differential equations, but it is of interest to apply to it some of the more recently developed techniques for the solution of such equations.[7] Fortunately, in the present case, we can turn to the energy equation as a first integral of the equation of motion and can integrate it in terms of a new variable.

Since the Coriolis force is at right angles to the velocity and does no work, the energy equation is simply

$$\tfrac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)+\tfrac{1}{2}mgr^2/l=\text{constant}. \quad (11)$$

At the maximum displacement A, the radial velocity $\dot{r}$ is zero and we can substitute from Eq. (9) and evaluate the constant. We find

$$\dot{r}^2+\omega^2 r^2+c^2/r^2=\omega^2 B^2, \quad (12)$$

where $B^2=A^2+c^2/\omega^2 A^2$. This equation can be written in terms of a new variable z, where $z=r^2$ and $\dot{z}=2r\dot{r}$, and becomes

$$\dot{z}^2=-4c^2+4\omega^2 B^2 z-4\omega^2 z^2. \quad (13)$$

This can be integrated and gives the solution of the nonlinear equation as

$$r^2=\tfrac{1}{2}B^2+\tfrac{1}{2}C^2\cos 2\omega t, \quad (14)$$

where $C^2=A^2-c^2/\omega^2 A^2$.

THE PRECESSION OF THE PENDULUM

The angular velocity $-\Omega\sin\phi$ required for harmonic oscillation along the radius vector is small. It is sometimes argued[3] that the bob may well acquire this angular velocity in starting from the rest position. That this is not the explanation of the regularly observed precession is seen by reflecting that a masking angular velocity c/A^2 is equally probable. The student must consider the effect of an arbitrary areal velocity c on the pendulum motion. Substituting from Eq. (14) in Eq. (9), we have

$$\dot{\theta}=2c/(B^2+C^2\cos 2\omega t)-\Omega\sin\phi. \quad (15)$$

This is integrable and, writing $\Delta\theta$ for the angle turned through in one complete oscillation of the pendulum, we have

$$\Delta\theta=-\Omega T\sin\phi. \quad (16)$$

Thus, the precession of the pendulum is shown to be $-\Omega\sin\phi$ in the anharmonic as in the harmonic case.

The general solution of the equations of motion enables us to calculate the position of the pendulum bob for any given initial conditions. In particular, from Eq. (14), we obtain the minimum displacement from the equilibrium position as $c/\omega A$. This gives at once the simple condition, $c=0$, that the bob pass through the equilibrium position. From Eq. (9), we see that under no condition can the Kepler law of equal areas be obeyed.

[7] See, for example, Schelkunoff, Quart. Appl. Math. **3**, 348 (1945).

Erratum: Quantitative Evaluation of Rocket Propellants

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EQUATION (12) should read

$$c^*=(1/\lambda)(RT_c/\delta\bar{M}^*)^{\frac{1}{2}}. \quad (12)$$