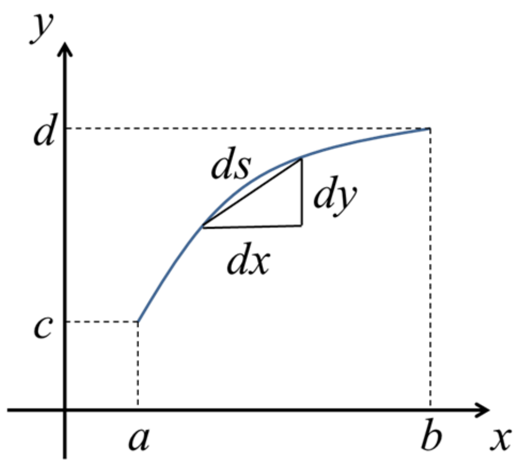
# §8-1 Arc Length

＊公式：弧長




$$l = \int ds$$

$$= \int \sqrt{(dx)^2 + (dy)^2}$$

$$= \begin{cases} \int_a^b \left( \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \right) dx = \int_a^b \left( \sqrt{1 + (f'(x))^2} \right) dx \\ \int_c^d \left( \sqrt{\left(\frac{dx}{dy}\right)^2 + 1} \right) dy = \int_c^d \left( \sqrt{(g'(y))^2 + 1} \right) dy \\ \int_{t_1}^{t_2} \left( \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \right) dt \text{ ; } x(t_1) = a \text{ ; } x(t_2) = b. \end{cases}$$



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**Example 1**： Evaluate the arc length of $y^2 = x^3$ from (0, 0) to $(\frac{1}{4}, \frac{1}{8})$.

**Solution**：

$$y = x^{\frac{3}{2}}$$

$$l = \int_0^{\frac{1}{4}} \sqrt{1 + \frac{9}{4}x}dx$$

$$= \frac{4}{9} \times \frac{2}{3}\left(1 + \frac{9}{4}x\right)^{\frac{3}{2}}\Bigg|_0^{\frac{1}{4}} = \frac{61}{216}.$$

**Example 2**： Evaluate the arc length of $y = x^2 - \frac{1}{8}\ln x$ from $(1, 1)$ to $(x, y)$.

**Solution**：

$$y' = 2x - \frac{1}{8x}$$

$$\Rightarrow l = \int_1^x \sqrt{1 + \left(2t - \frac{1}{8t}\right)} dt$$

$$= t^2 + \frac{1}{8}\ln|t|\Bigg|_1^x$$

$$= x^2 + \frac{1}{8}\ln|x| - 1.$$

**Example 3**： Find the length of the curve $y=\int_1^x \left(\sqrt{t^3-1}\right)dt, 1 \le x \le 4.$

**Solution**：

$$y'=\sqrt{x^3-1}$$

$$\Rightarrow l=\int_1^4 \sqrt{1+x^3-1}dx$$

$$=\frac{2}{5}x^{\frac{5}{2}}\Big|_1^4=\frac{62}{5}.$$

**Example 4**：Prove that the circumference of the circle with radius $r$ is $2\pi r$.

**Prove**：

$$y=f(x)=\sqrt{r^2-x^2}\Rightarrow f'(x)=-x\left(r^2-x^2\right)^{-\frac{1}{2}}$$

$$\Rightarrow 1+\left(y'\right)^2=\frac{r^2}{r^2-x^2}$$

$$\Rightarrow \text{圓周長}=4r\int_0^r \frac{1}{r\cos\theta}r\cos\theta\, d\theta$$

$$=2\pi r.$$

歷屆大會考考古題精選：

1. Let $L(C_i)$ represent the length of curve $C_i$ ; $1 \le i \le 4$. Which curve has the longest length?

(A) $C_1$ (B) $C_2$ (C) $C_3$ (D) $C_4$

With

$$L(C_1) = \int_0^1 4\sqrt{1 + x^2\left(1 - x^2\right)^{-1}}\,dx$$

$$L(C_2) = \int_0^{2\pi} \sqrt{\sin^2\theta + 4\cos^2\theta}\,d\theta$$

$$L(C_3) = \int_0^{4\pi} \frac{1}{2}\sqrt{\cos^2\frac{\theta}{2} + \frac{1}{4}\sin^2\frac{\theta}{2}}\,d\theta$$

$$L(C_4) = \int_0^{\sqrt{2}} \sqrt{\left(2t\right)^2 + \left(2t\right)^2}\,dt.$$

2. Length of the curve $y = e^x$, $0 \le x \le 1$ is

(A) $\sqrt{1+e^2} - \sqrt{2} + \ln\left(\sqrt{1+e^2} - 1\right) - 1 - \ln\left(\sqrt{2} - 1\right)$.

(B) $\sqrt{1+e^2} - \sqrt{2} - \ln\left(\sqrt{1+e^2} - 1\right) - 1 + \ln\left(\sqrt{2} - 1\right)$.

(C) $\sqrt{1+e^2} + \sqrt{2} - \ln\left(\sqrt{1+e^2} - 1\right)$.

(D) -$\sqrt{1+e^2} + \sqrt{2} + \ln\left(\sqrt{1+e^2} - 1\right)$.

3. The length of the curve with equation $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$ is __________.



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