

§7-8 Improper Integrals

* What is the improper integral $\int_a^b f(x)dx$?

1. $a = \pm\infty$ or $b = \pm\infty$.
2. $f(c) = \pm\infty$ for some $c \in [a, b]$.

發生以上 1. 或 2. 情況，即可稱 $\int_a^b f(x)dx$ 為 improper integral.

* 幾何上來說，improper integral 所圍出來的區域是無界的。

$$(1-a) \quad \int_a^\infty f(x)dx = \lim_{b \rightarrow \infty} \int_a^b f(x)dx$$

$$(1-b) \quad \int_{-\infty}^\infty f(x)dx = \int_{-\infty}^0 f(x)dx + \int_0^\infty f(x)dx$$

(1-a) (1-a)

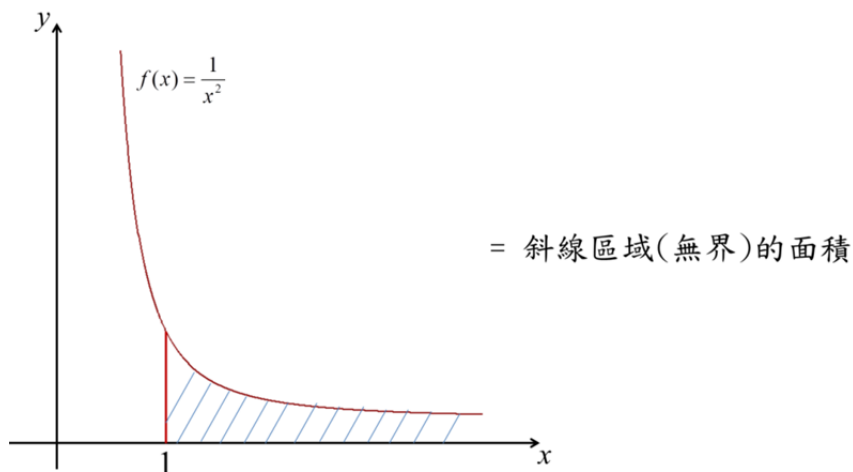
$$(2-a) \quad c = a \text{ or } b$$

$$\int_a^b f(x)dx = \lim_{d \rightarrow a^+} \int_d^b f(x)dx \text{ or } \int_a^b f(x)dx = \lim_{d \rightarrow b^-} \int_a^d f(x)dx$$

$$(2-b) \quad a < c < b$$

$$\int_a^b f(x)dx = \lim_{d \rightarrow c^-} \int_a^d f(x)dx + \lim_{d \rightarrow c^+} \int_d^b f(x)dx$$

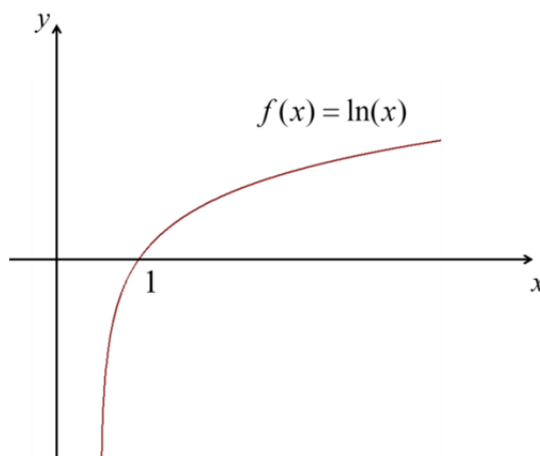
* c means the discontinuous points.



* 一個 improper Integral 可能有好幾個缺點

$$\int_0^{\infty} \ln x dx = \int_0^1 \ln x dx + \int_1^{\infty} \ln x dx$$

$$= \lim_{b \rightarrow 0^+} \int_b^1 \ln x dx + \lim_{c \rightarrow \infty} \int_1^c \ln x dx$$



* 註記：

1. 要計算具有多個缺點的 improper integral，一定要將原 improper integral 表成多個新的 improper integrals 的組合，其中每一個新的 improper integral 都只能有一個缺點。
2. 如果每一個只有一個缺點的 improper integral 所代表的無界區域之面積皆為有界，即原 improper integral 稱之為收斂 (convergence)，反之則稱為發散 (divergence)。

Example 1 : Verify the convergence of $\int_1^{\infty} \frac{1}{x^p} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x^p} dx$.

Solution :

$$\int_1^b \frac{1}{x^p} dx = \begin{cases} \frac{x^{-p+1}}{-p+1} \Big|_1^b & p \neq 1 \\ \ln |x| \Big|_1^b & p = 1 \end{cases}$$

$$\Rightarrow \int_1^{\infty} \frac{1}{x^p} dx = \begin{cases} \text{convergence} & p > 1 \\ \text{divergence} & p \leq 1 \end{cases}$$

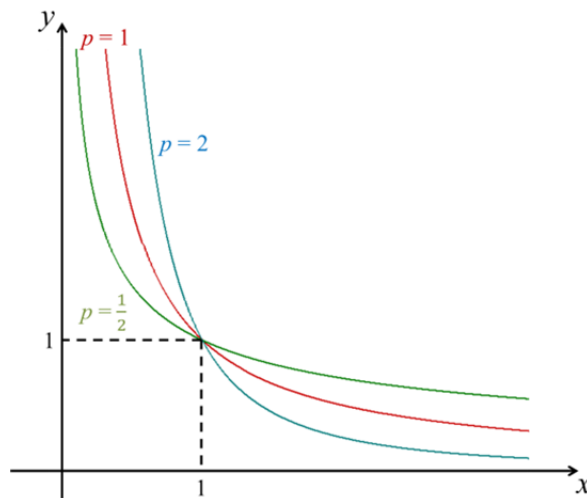
Example 2 : Verify the convergence of $\int_0^1 \frac{1}{x^p} dx = \lim_{b \rightarrow 0^+} \int_b^1 \frac{1}{x^p}$.

Solution :

$$\int_0^1 \frac{1}{x^p} dx = \begin{cases} \text{convergence} & p < 1 \\ \text{divergence} & p \geq 1 \end{cases}$$

*Summary :

- i. Example 1 中， p 要大(即 $f(x) = \frac{1}{x^p}$ 跑到 0 的速度要夠快，當 $x \rightarrow \infty$ 時)，無窮區域的面積才可能是有限。
- ii. Example 2 中， p 要小(即 $f(x) = \frac{1}{x^p}$ 跑到 ∞ 的速度要夠慢，當 $x \rightarrow 0$ 時)，無窮區域的面積才有可能有限。



*Comparison Theorem : $f \geq g \geq 0$

- i. $\int_a^b g(x)dx : \text{diverges} \Rightarrow \int_a^b f(x)dx : \text{diverges}.$
- ii. $\int_a^b f(x)dx : \text{converges} \Rightarrow \int_a^b g(x)dx : \text{converges}.$

Example 3 : Verify the convergence of $\int_{-\infty}^0 xe^x dx$.

Solution :

Converges.

Example 4 : Verify the convergence of $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$.

Solution :

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx \\ &= \int_{-\infty}^0 \frac{1}{1+x^2} dx + \int_0^{\infty} \frac{1}{1+x^2} dx \\ &= \tan^{-1} x \Big|_{-\infty}^0 + \tan^{-1} x \Big|_0^{\infty} \\ &= \frac{\pi}{2} + \frac{\pi}{2} = \pi. \end{aligned}$$

Example 5 : Verify the convergence of $\int_0^3 \frac{1}{x-1} dx$.

Solution :

$$\begin{aligned} & \int_0^3 \frac{1}{x-1} dx \\ &= \int_0^1 \frac{1}{x-1} dx + \int_1^3 \frac{1}{x-1} dx. \\ & \quad \text{diverges} \qquad \text{diverges} \end{aligned}$$

Example 6 : Verify the convergence of $\int_0^1 \ln x dx$.

Solution :

$$\begin{aligned} & \int_0^1 \ln x dx \\ &= x \ln x \Big|_0^1 - x \Big|_0^1 = -1 \end{aligned}$$

* 另解： $-\ln x$ 比 $\frac{1}{x^2}$ 跑到 ∞ 的速度慢，當 $x \rightarrow 0$ 時 \Rightarrow converges.

Example 7 : Verify the convergence of $\int_2^\infty \frac{1}{\ln x} dx$.

Solution :

$$\because \frac{1}{x} < \frac{1}{\ln x} \text{ 當 } x \text{ 很大時 } \Rightarrow \text{diverges.}$$

Example 8 : Verify the convergence of $\int_0^\infty e^{-x^2} dx$.

Solution :

$$e^{x^2} > x^2 \Rightarrow e^{-x^2} < x^{-2} \Rightarrow \text{converges. (compare with } \frac{1}{x^2} \text{)}$$

Example 9 : Verify the convergence of $\int_1^\infty \frac{1+e^{-x}}{x} dx$.

Solution :

$$\frac{1+e^{-x}}{x} > \frac{1}{x} \Rightarrow \text{diverges. (compare with } \frac{1}{x} \text{)}$$

Example 10 : Verify the convergence of $\int_0^{\infty} x^2 e^{-x^2} dx$.

Solution :

$$u = x \quad dv = x e^{-x^2} dx$$

$$du = dx \quad v = -\frac{1}{2} e^{-x^2}$$

$$\int_0^{\infty} x^2 e^{-x^2} dx$$

$$= -\frac{1}{2} x e^{-x^2} \Big|_0^{\infty} + \frac{1}{2} \int_0^{\infty} e^{-x^2} dx$$

$$= \frac{1}{2} \int_0^{\infty} e^{-x^2} dx.$$

(和 $\frac{1}{x^2}$ 比較可得此瑕積分為收斂)

歷屆大會考考古題精選：

1. If n is a positive integer, then $\int_0^{\infty} x^n e^{-x} dx = ?$

(A) 1; (B) n ; (C) $n!$; (D) ∞ .

2. Evaluate $\int_1^{\infty} \frac{\ln x}{x^2} dx = \underline{\hspace{2cm}}$.

3. The value of $\int_0^{\infty} \frac{\ln x}{1+x^2} dx$ is $\underline{\hspace{2cm}}$.

4. Evaluate the improper integral $\int_0^1 \frac{\ln x}{\sqrt{x}} dx = \underline{\hspace{2cm}}$.

5. Evaluate $\int_{-\infty}^{\infty} \frac{dx}{4x^2 + 4x + 5} = \underline{\hspace{2cm}}$.