§7-4 Integration of Rational Function by Partial Fraction

How to integrate $\int \frac{P(x)}{Q(x)}dx$? Here $P(x)$, $Q(x)$ are polynomials?

Step1：Write $\frac{P(x)}{Q(x)}$ as $\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$, where deg R < deg Q.

Step2：將 $Q(x)$ 因式分解成一次或兩次的連乘積，即 $Q(x)$的因式基本型為 $(ax^2 + bx + c)^n$，和$(ex + f)^m$, where $b^2 - 4ac < 0$, and n & m are nonnegative integers.

Step3：

i. 含$(ex + f)^m$的因式，將 $\frac{R(x)}{Q(x)}$ 寫成 partial fractions(部份分式)時，需含有下列各項：

$$(ex+f)^m \Rightarrow \frac{f_1}{ex+f} + \frac{f_2}{(ex+f)^2} + \frac{f_3}{(ex+f)^3} + ... + \frac{f_m}{(ex+f)^m} \text{ ……(1)}$$

ii. 同理，含$(ax^2 + bx + c)^n$的因式，部份分式需含下列各項：

$$(ax^2+bx+c)^n \Rightarrow \frac{b_1x+c_1}{ax^2+bx+c} + \frac{b_2x+c_2}{(ax^2+bx+c)^2} + ... + \frac{b_nx+c_n}{(ax^2+bx+c)^n} \text{ ……..(2)}$$

＊Why?

給一$(m-1)$次多項式 $R(x)$，可將 $R(x)$唯一的表成

$$R(x) = f_1(ex+f)^{m-1} + f_2(ex+f)^{m-2} + ... + f_{m-1}(ex+f) + f_m \text{ (why?)}$$

(即將 x 轉成 $ex + f$)

$$\Rightarrow \frac{R(x)}{(ex+f)^m} = \frac{f_1}{ex+f} + \frac{f_2}{(ex+f)^2} + ... + \frac{f_{m-1}}{(ex+f)^{m-1}} + \frac{f_m}{(ex+f)^m}.$$

Step4：

i. (1)式每一項都很容易逐項積分。

國立交通大學應用數學系 莊重教授

ii. (2)式每一項都可以透過配方法及 substitution 寫成

$$\frac{\alpha x+\beta}{\left(u^2+r\right)^i}=\underbrace{\frac{\alpha u}{\left(u^2+r\right)^i}}_{\text{Direct substitution}}+\underbrace{\frac{\beta}{\left(u^2+r\right)^i}}_{\text{Trigonometric substitution}}$$

Example 1：$\int\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}dx=?$

Solution：

$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\overset{\text{長除法}}{=}(x+1)+\frac{4x}{(x+1)\times\left(x-1\right)^2}$$

$$\frac{R(x)}{Q(x)}=\frac{4x}{(x+1)\times\left(x-1\right)^2}=\frac{a}{x+1}+\frac{b}{x-1}+\frac{c}{\left(x-1\right)^2}$$

$$\Rightarrow 4x=a\times\left(x-1\right)^2+b\times(x+1)(x-1)+c\times(x+1)$$

$x=-1 \quad\Rightarrow\quad a=-1$

$x=1 \quad\Rightarrow\quad c=2$

比較2次項係數：$0=a+b\Rightarrow b=1$

$$\Rightarrow \text{原式}=\int(x+1)dx+\int\left(\frac{-1}{x+1}\right)+\frac{1}{x-1}+\frac{2}{\left(x-1\right)^2}dx$$

$$=\frac{x^2}{2}+x-\ln|x+1|+\ln|x-1|-\frac{2}{(x-1)}+c.$$

Example 2：$\int \frac{2x^2-x+4}{x^3+4x}dx=?$

Solution：

$$\frac{2x^2-x+4}{x^3+4x}=\frac{2x^2-x+4}{x\left(x^2+4\right)}=\frac{a}{x}+\frac{bx+c}{x^2+4}$$

$$\Rightarrow 2x^2-x+4=a\times\left(x^2+4\right)+x\times(bx+c)$$

$$x=0\Rightarrow a=1$$

比較係數：$\begin{cases}a+b=2\\c=-1\end{cases}\Rightarrow b=1$

$$\Rightarrow \text{原式}=\int\frac{1}{x}+\frac{x-1}{x^2+4}dx$$

$$=\int\left(\frac{1}{x}+\frac{x}{x^2+4}-\frac{1}{x^2+4}\right)dx$$

$$=\ln|x|+\frac{1}{2}\ln\left(x^2+4\right)-\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+c.$$

Example 3：$\int \frac{4x^2-3x+2}{4x^2-4x+3}dx=?$

Solution：

$$\int\frac{4x^2-3x+2}{4x^2-4x+3}dx$$

$$=\int 1dx+\int\frac{x+1}{\left(2x-1\right)^2+2}dx \qquad \begin{pmatrix}u=2x-1\Rightarrow du=2dx\\x=\frac{u+1}{2}\end{pmatrix}$$

$$= x + \frac{1}{2}\int \frac{\frac{u+1}{2} - 1}{u^2 + 2}\, du$$

$$= x + \frac{1}{4}\int \frac{u-1}{u^2+2}\, du$$

$$= x + \frac{1}{8}\ln\left(u^2+2\right) - \frac{1}{4\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}} + c.$$

Example 4：$\int \frac{1 - x + 2x^2 - x^3}{x\left(x^2+1\right)^2}\, dx = ?$

Solution：

$$\frac{1 - x + 2x^2 - x^3}{x\left(x^2+1\right)^2} = \frac{a}{x} + \frac{bx + c}{x^2+1} + \frac{dx + e}{\left(x^2+1\right)^2}$$

$$\Rightarrow \left(1 - x + 2x^2 - x^3\right) = a\left(x^2+1\right)^2 + x\left(bx + c\right) \times \left(x^2+1\right) + x\left(dx + e\right)$$

$x = 0 \Rightarrow a = 1$

比較係數：$\begin{cases} 0 = a + b & \Rightarrow b = -1 \\ -1 = c & \\ 2 = 2a + b + d & \Rightarrow e = 0 \\ -1 = c + f & \Rightarrow d = 1 \end{cases}$

$$\Rightarrow \text{原式} = \ln|x| - \frac{1}{2}\ln\left(x^2+1\right) - \tan^{-1} x - \frac{1}{2\left(x^2+1\right)} + c.$$

Example 5：$\int \frac{1}{x\sqrt{x+1}}dx = ?$

Solution：

Let $u = (x+1)^{\frac{1}{2}} \Rightarrow u^2 = x+1 \Rightarrow 2udu = dx$

$$\int \frac{1}{x\sqrt{x+1}}dx$$

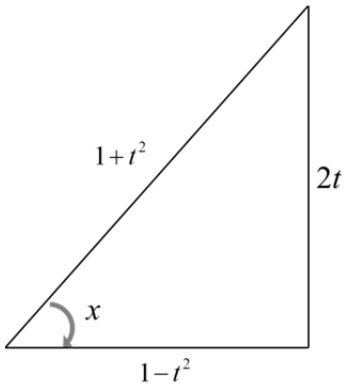
$$= \int \frac{2udu}{u\left(u^2-1\right)}$$

$$= \int \left(\frac{1}{u-1} - \frac{1}{u+1}\right)du$$

$$= \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + c.$$

Example 6：$\int \frac{1}{3\sin x - 4\cos x}\,dx = ?$

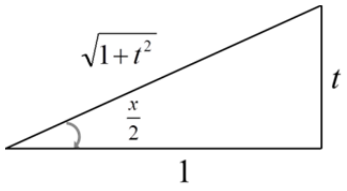
Solution：



$$t = \tan\frac{x}{2} \Rightarrow \tan x = \tan\left(\frac{x}{2}+\frac{x}{2}\right) = \frac{2t}{1-t^2}$$

$$\Rightarrow \sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$$

$$dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx = \frac{1}{2}\left(1+t^2\right)dx$$



$$\Rightarrow dx = \frac{2}{1+t^2}\,dt$$

$$\text{原式} = \int \frac{P(t)}{Q(t)}\,dt = \int \frac{1}{2t^2+3t-2}\,dt$$

即任一rational function of $\sin x$ & $\cos x$

可改成rational function of t, $t = \tan\frac{x}{2}$.