

§7-3 Trigonometric Substitution

*題型：(利用三角代換法將根號除掉)

$$1. \sqrt{a^2 - x^2} \xrightarrow{x=a \sin \theta} a \cos \theta, \quad \theta \in I \cup IV.$$

$$2. \sqrt{x^2 - a^2} \xrightarrow{x=a \sec \theta} a \tan \theta, \quad \theta \in I \cup III.$$

$$3. \sqrt{x^2 + a^2} \xrightarrow{x=a \tan \theta} a \sec \theta, \quad \theta \in I \cup IV.$$

Example 1 : $\int \frac{\sqrt{9-x^2}}{x^2} dx = ?$

Solution :

$$\text{Let } x = 3 \sin \theta \Rightarrow dx = 3 \cos \theta d\theta$$

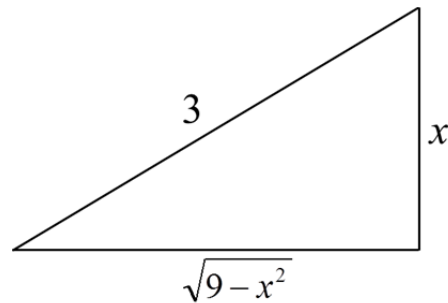
$$\int \frac{\sqrt{9-x^2}}{x^2} dx$$

$$= \int \frac{3 \cos \theta}{9 \sin^2 \theta} d\theta$$

$$= \int \frac{1 - \sin^2 \theta}{\sin^2 \theta} d\theta$$

$$= -\cot \theta - \theta + c$$

$$= -\frac{\sqrt{9-x^2}}{x} - \sin^{-1} \frac{x}{3} + c.$$



Example 2 : $\int \frac{1}{x^2 \sqrt{x^2 + 4}} dx = ?$

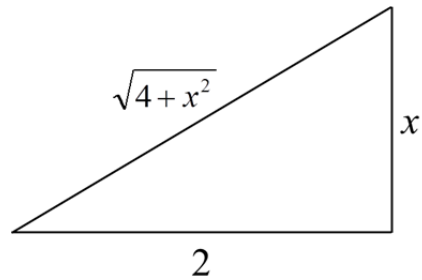
Solution :

$$\text{Let } x = 2 \tan \theta \Rightarrow dx = 2 \sec^2 \theta d\theta$$

$$\begin{aligned} & \int \frac{1}{x^2 \sqrt{x^2 + 4}} dx \\ &= \int \frac{2 \sec^2 \theta}{4 \tan^2 \theta 2 \sec \theta} d\theta \\ &= \frac{1}{4} \int \frac{\sec \theta}{\tan^2 \theta} d\theta \\ &= \frac{1}{4} \int \frac{\cos \theta}{\sin^2 \theta} d\theta \end{aligned}$$

$$\text{Let } \sin \theta = u \Rightarrow du = \cos \theta d\theta$$

$$\begin{aligned} & \Rightarrow \frac{1}{4} \int \frac{1}{u^2} du \\ &= -\frac{1}{4} \frac{1}{u} + c \\ &= -\frac{1}{4} \frac{1}{\sin \theta} + c \\ &= -\frac{1}{4} \frac{\sqrt{x^2 + 4}}{x} + c. \end{aligned}$$



Example 3 : $\int \frac{x}{\sqrt{x^2+4}} dx = ?$

Solution :

$$\begin{aligned} & \int \frac{x}{\sqrt{x^2+4}} dx \\ &= (x^2+4)^{\frac{1}{2}} + c. \end{aligned}$$

* 註記：若 substitution 的方法可解決，優先使用。

Example 4 : $\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{(4x^2+9)^{\frac{3}{2}}} dx = ?$

Solution :

$$\text{Let } x = \frac{3}{2} \tan \theta \Rightarrow dx = \frac{3}{2} \sec^2 \theta d\theta$$

$$\begin{aligned} & \int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{(4x^2+9)^{\frac{3}{2}}} dx \\ &= \int_0^{\frac{\pi}{3}} \frac{\frac{27}{8} \tan^3 \theta \frac{3}{2} \sec^2 \theta}{27 \sec^3 \theta} d\theta \\ &= \frac{3}{16} \int_0^{\frac{\pi}{3}} \frac{\tan^3 \theta}{\sec \theta} d\theta \\ &= \frac{3}{16} \int_0^{\frac{\pi}{3}} \frac{\sin^3 \theta}{\cos^2 \theta} d\theta \\ &= \frac{3}{16} \int_0^{\frac{\pi}{3}} \frac{(1-\cos^2 \theta)}{\cos^2 \theta} \sin \theta d\theta \end{aligned}$$

$$= -\frac{3}{16} \int_1^{\frac{1}{2}} \frac{1-u^2}{u^2} du \quad (u = \cos \theta)$$

$$= \frac{3}{16} \int_{\frac{1}{2}}^1 (u^{-2} - 1) du$$

$$= \frac{3}{16} \left(-u^{-1} - u \Big|_{\frac{1}{2}}^1 \right) = \frac{3}{32}.$$

Example 5 : $\int \frac{x}{\sqrt{3-2x-x^2}} dx = ?$

Solution :

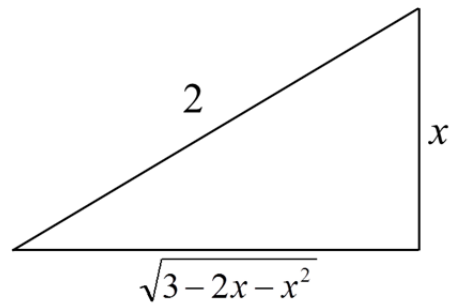
$$\int \frac{x}{\sqrt{3-2x-x^2}} dx = \int \frac{x}{\sqrt{-(x+1)^2 + 4}} dx$$

Let $x = -1 + 2 \sin \theta \Rightarrow dx = 2 \cos \theta d\theta$

$$\Rightarrow \int \frac{-1 + 2 \sin \theta}{2 \cos \theta} 2 \cos \theta d\theta$$

$$= -\theta - 2 \cos \theta + c$$

$$= -\sin^{-1} \frac{x+1}{2} - \sqrt{3-2x-x^2} + c.$$



Example 6 : $\int_0^{\frac{\pi}{2}} \frac{\cos t}{\sqrt{1+\sin^2 t}} dt = ?$

Solution :

Let $\sin t = u \Rightarrow \cos t dt = du$

$$\Rightarrow \int_0^1 \frac{1}{\sqrt{1+u^2}} du$$

Let $u = \tan \theta \Rightarrow du = \sec^2 \theta d\theta$

$$\Rightarrow \int_0^{\frac{\pi}{4}} \sec \theta d\theta = \ln|\sec \theta + \tan \theta| \Big|_0^{\frac{\pi}{4}} = \ln(\sqrt{2} + 1)$$

Example 7 :

$$\int \sqrt{\frac{a+cx}{x}} dx = ? \text{ (Hint : 令 } x = ? \text{ 可以將 } \sqrt{\quad} \text{ 消掉?)}$$