§7-1 Integration by Parts

＊公式：Integration by parts

- $\int u dv = uv - \int v du.$
- $\int_a^b u dv = uv\Big|_a^b - \int_a^b v du.$

＊Integration by parts (積分) ↔ Product rule (微分)

$u = f(x), v = g(x);$

$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$

$\Rightarrow \int \underbrace{(f(x)g(x))'}_{uv} dx = \int \underbrace{g(x)}_{v} \underbrace{f'(x)}_{du} dx + \int \underbrace{f(x)}_{u} \underbrace{g'(x)}_{dv} dx$

$\Rightarrow uv = \int v du + \int u dv$

$\Rightarrow \int u dv = uv - \int v du.$

＊Keys：

1. Identify u and dv.

2. Easy to recover v from dv.

3. $\int vdu$ is not complicated than $\int udv$.

Example 1： $\int x \sin x dx = ?$

Solution：

$$\begin{cases} u = x \\ dv = \sin x dx \end{cases} \Rightarrow \begin{cases} du = dx \\ v = -\cos x \end{cases}$$

$$\Rightarrow \int x \sin x dx = -x\cos x + \int \cos x dx = -x\cos x + \sin x + c.$$

Example 2： $\int \tan^{-1} x dx = ?$

Solution：

$$\begin{cases} u = \tan^{-1} x \\ dv = dx \end{cases} \Rightarrow \begin{cases} du = \dfrac{dx}{1+x^2} \\ v = x \end{cases}$$

$$\Rightarrow \int \tan^{-1} x dx = x\tan^{-1} x - \int \frac{x}{1+x^2} dx = x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + c.$$

Example 3： $\int \ln(x-1) dx = ?$

Solution：

$$\begin{cases} u = \ln(x-1) \\ dv = dx \end{cases} \Rightarrow \begin{cases} du = \dfrac{dx}{x-1} \\ v = x-1 \end{cases}$$

$$\Rightarrow \int \ln(x-1) dx = (x-1)\ln(x-1) - \int 1 dx = (x-1)\ln(x-1) - x + c.$$

Example 4： $\int x^2 e^x dx = ?$

Solution：

$$\begin{cases} u = x^2 \\ dv = e^x dx \end{cases} \Rightarrow \begin{cases} du = 2x dx \\ v = e^x \end{cases}$$

$$\Rightarrow \int x^2 e^x dx = x^2 e^x - \underline{\int 2x e^x dx}$$

$$\begin{cases} u = 2x \\ dv = e^x dx \end{cases} \Rightarrow \begin{cases} du = 2dx \\ v = e^x \end{cases}$$

$$\int x^2 e^x dx = x^2 e^x - \left(\underline{2xe^x - 2\int e^x dx} \right) = x^2 e^x - 2xe^x + 2e^x + c$$

Example 5：$\int e^x \sin x dx = ?$

Solution：

$$\begin{cases} u = \sin x \\ dv = e^x dx \end{cases} \Rightarrow \begin{cases} du = \cos x dx \\ v = e^x \end{cases}$$

$$\int e^x \sin x dx = e^x \sin x - \int \underline{e^x \cos x dx}$$

$$\begin{cases} u = \cos x \\ dv = e^x dx \end{cases} \Rightarrow \begin{cases} du = -\sin x dx \\ v = e^x \end{cases}$$

$$\int e^x \sin x dx = e^x \sin x - \left(\underline{e^x \cos x + \int e^x \sin x dx} \right)$$

$$\Rightarrow \int e^x \sin x dx = \frac{1}{2} e^x (\sin x - \cos x) + c$$

Example 6：$\int \cos \sqrt{x} dx = ?$ (Hint：Substitution + Integration by parts)

Solution：

Let $u = \sqrt{x} \Rightarrow du = \frac{1}{2} x^{-\frac{1}{2}} dx$

$$\Rightarrow 2\int u \cos u du$$

$$\begin{cases} v = u \\ dw = \cos u du \end{cases} \Rightarrow \begin{cases} dv = du \\ w = \sin u \end{cases}$$

$$\Rightarrow 2u \sin u - \int \sin u du$$

$$= 2u \sin u + \cos u + c$$

$$= 2\sqrt{x} \sin \sqrt{x} + \cos \sqrt{x} + c.$$

Example 7：$\int_0^1 x^3 e^{-x^2} dx = ?$

Solution：

Let $u = x^2 \Rightarrow du = 2xdx$

$$\Rightarrow \frac{1}{2}\int_0^1 ue^{-u} du = \frac{1}{2}\left(-ue^{-u} - e^{-u}\Big|_0^1\right) = \frac{1}{2}\left[(-e^{-1} - e^{-1}) + 1\right] = -e^{-1} + \frac{1}{2}.$$