§6-2─6-3 Volumes by Cylindrical Shells

＊公式：

$$V_3 = \int V_2 dx$$

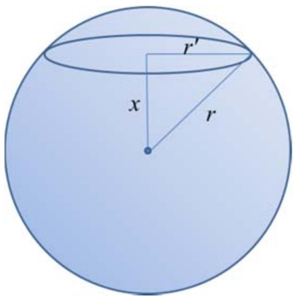
$V_3 = \begin{cases} \text{ } \\ \text{ } \end{cases}$


§6-2(截面為圓環)


§6-3(截面為圓柱的表面)

Example 1：Prove that the volume of the sphere with radius r is $\frac{4}{3}\pi r^3$.

Solution：

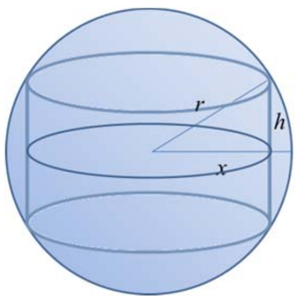


$$V = 2\int_0^r \pi r'^2\, dx = 2\int_0^r \pi(r^2 - x^2)dx$$

$$= \frac{4}{3}\pi r^3$$

§6-2

$$V = 2\int_0^r 2\pi \times xh dx$$



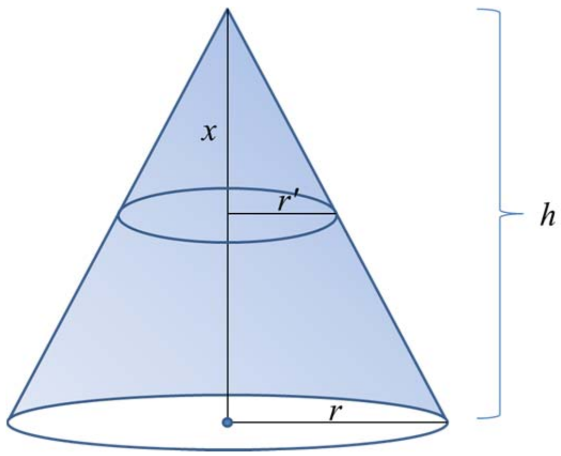
$$= 2\int_0^r 2\pi \times x\sqrt{r^2 - x^2}\, dx$$

$$= -\frac{4\pi}{3}\left(r^2 - x^2\right)^{\frac{3}{2}}\Big|_0^r$$

$$= \frac{4}{3}\pi r^3$$

§6-3

Example 2：Find the volume of the following graph.



Solution：

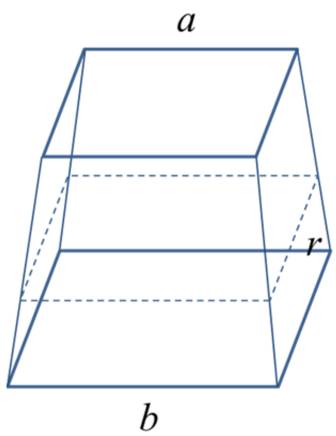
$$\frac{x}{h}=\frac{r'}{r}$$

$$\Rightarrow r'=\frac{r}{h}x.$$

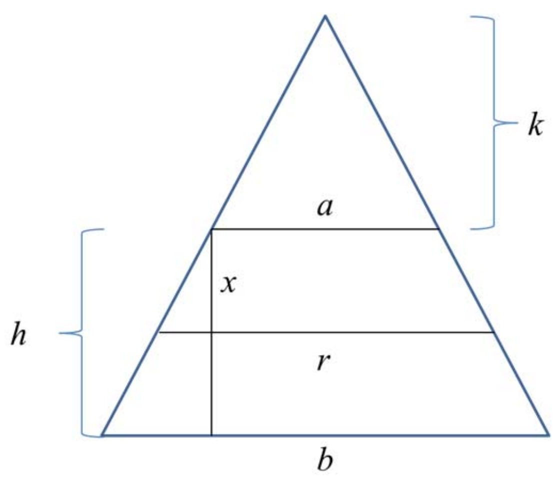
$$V=\int_0^h \pi r'^2\,dx=\int_0^h \pi\left(\frac{r^2}{h^2}\right)x^2dx$$

$$=\frac{1}{3}\pi r^2 h.$$

Example 3：A frustum of a pyramid with square base of side b, square top of side a, and height h.



Solution：



$$V=\int_0^h r^2dx$$

$$\frac{k}{h}=\frac{a}{b-a}\Rightarrow k=\frac{ah}{b-a}$$

$$\frac{x+k}{k}=\frac{r}{a}\Rightarrow r=a+\left(\frac{b-a}{h}\right)x$$

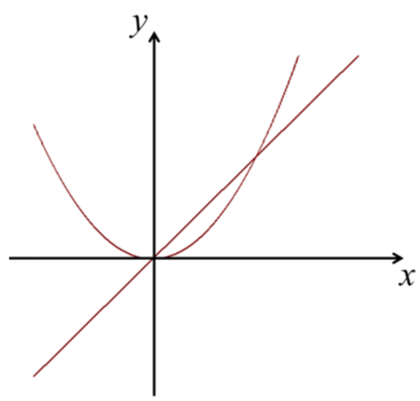
$$\Rightarrow V=\int_0^h\left(a+\frac{b-a}{h}x\right)^2dx$$

$$=\frac{h}{3(b-a)}\left(a+\frac{b-a}{h}x\right)^3\Bigg|_0^h$$

$$=\frac{h}{3}\left(a^2+ab+b^2\right).$$

Example 4： Rotating $\begin{cases} y=x \\ y=x^2 \end{cases}$ about the x-axis.

Solution：



Method 1：(截面為環狀)

$$V=\int_0^1(\pi r_1^2-\pi r_2^2)dx,\quad r_1=x, r_2=x^2.$$

$$=\int_0^1(\pi x^2-\pi x^4)dx=\frac{2\pi}{15}$$

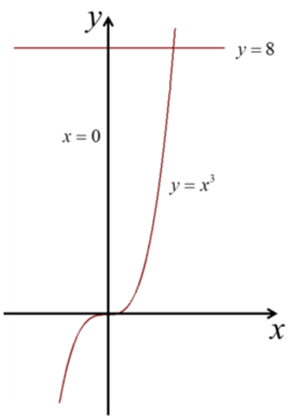
Method 2：(截面為圓柱狀)

$$V=\int_0^1 2\pi rh\,dy$$

$$=\int_0^1 2\pi y\left(\sqrt{y}-y\right)dy.$$

Example 5：Rotating $\begin{cases} y=8 \\ y=x^3 \\ x=0 \end{cases}$ about the x-axis.

Solution：



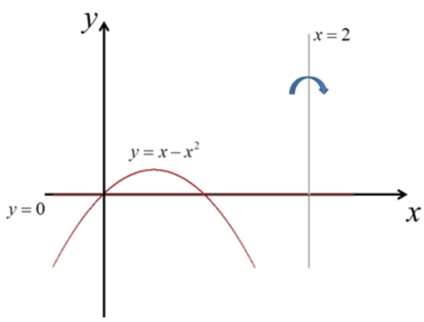
$$V=\int_0^2\left(64-x^6\right)\pi dx(\text{截面為環狀})$$

$$=\int_0^8 2\pi y^{\frac{1}{3}}y\,dy(\text{截面為圓柱狀})$$

$$=\frac{768}{7}\pi.$$

Example 6：Rotating $\begin{cases} y=x-x^2 \\ y=0 \end{cases}$ about the line $x=2$.

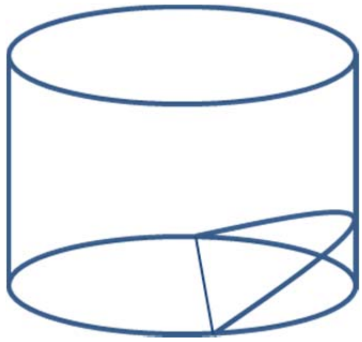
Solution：



$$V=\int_0^1 2\pi(2-x)(x-x^2)dx$$

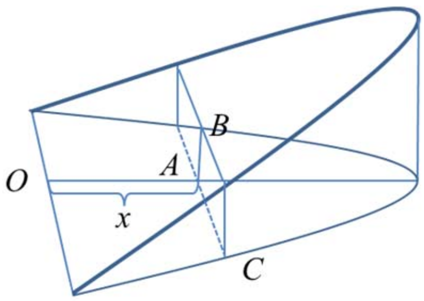
$$=\frac{\pi}{2}.$$

Example 7： A wedge is cut out of a circular cylinder of radius 4 by two planes. One plane is perpendicular to the axis of the cylinder. The other intersects the first at an angle of 30° along a diameter of the cylinder. Find the volume of the wedge.



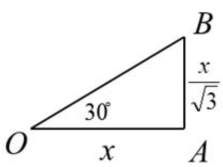
Solution：

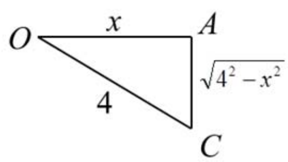
Method 1：



$$V=\int_0^4 \text{長方形截面}\,dx$$

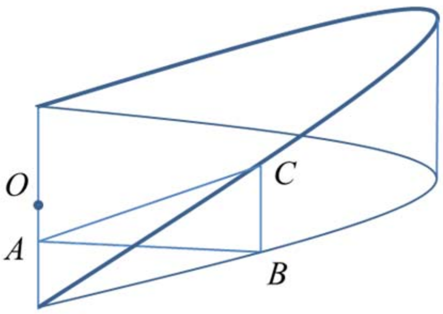
$$=\int_0^4 \underbrace{2\sqrt{4^2-x^2}}_{\text{長}}\underbrace{\left(\frac{x}{\sqrt{3}}\right)}_{\text{寬}}dx$$





$$=\frac{128}{3\sqrt{3}}.$$

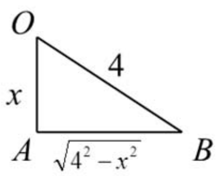
Method 2：

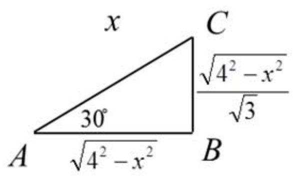


$$V=\int_0^4 \Delta\text{ABC截面}\,dx$$

$$=\int_0^4 \overline{AB}\times\overline{BC}\,dx$$

$$=\int_0^4 \frac{16-x^2}{\sqrt{3}}dx$$

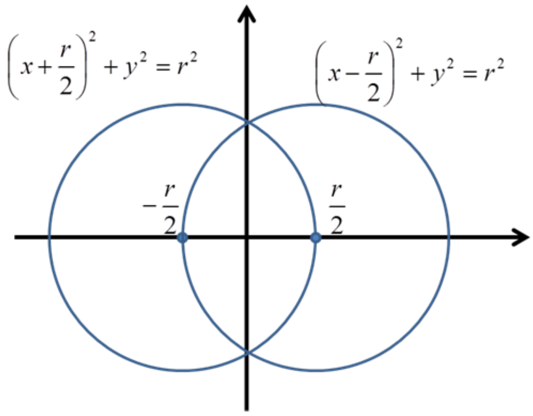




$$=\frac{128}{3\sqrt{3}}.$$

Example 8： Find the volume common to two spheres, each with radius r. If the center of each sphere lies on the surface of other.

Solution：



Method 1：

$$V=2\int_0^{\frac{r}{2}}\pi y^2dx$$

$$=2\int_0^{\frac{r}{2}}\pi\left(r^2-\left(x+\frac{r}{2}\right)^2\right)dx$$

$$=\frac{5}{12}\pi r^3.$$

Method 2：

$$V=2\int_{-\frac{r}{2}}^{0}\pi y^2dx$$

$$=2\int_{-\frac{r}{2}}^{0}\pi\left(r^2-\left(x-\frac{r}{2}\right)^2\right)dx$$

$$=\frac{5}{12}\pi r^3.$$

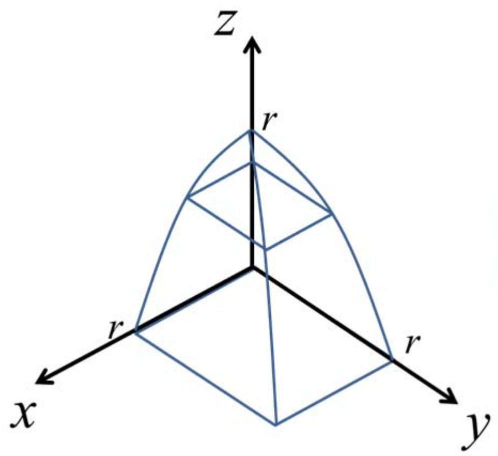
Example 9：Find the volume common to two circular cylinders, each with radius r. If the axes of the cylinders intersect at right angle.

＊想一想：

1. 先想想看，交集區域或是交集區域集面樣子？

2. 若想不出來，試著用代數手法組成此交集區域。

Solution：



$$\begin{cases} x^2+z^2 \le r^2 \\ y^2+z^2 \le r^2 \end{cases}$$ (circular cylinders)

由對稱性，我們只考慮交集的$\frac{1}{8}$區域，即 $x \ge 0, y \ge 0, z \ge 0.$

- 當 $z=0 \Rightarrow 0 \le x \le r, 0 \le y \le r$

  (正方形，邊長 $=r$)

- 當 $z=r \Rightarrow x=0, y=0$

  (正方形，邊長 $=0$)

- 當 $0<z<r \Rightarrow 0 \le x \le \sqrt{r^2-z^2}, 0 \le y \le \sqrt{r^2-z^2}$

  (正方形，邊長$=\sqrt{r^2-z^2}$)

⇒當 z 固定，$0 \le z \le r$，交集區域的橫截面為正方形。

⇒交集區域的體積$V=8\int_0^r \left(r^2-z^2\right)dz=\frac{16}{3}r^3.$