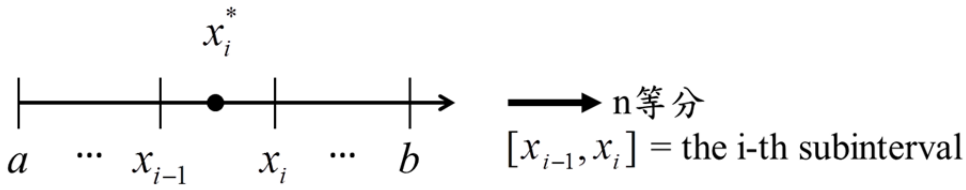
§5-2 The Definite Integral

＊本章將介紹定積分 $\int_a^b f(x)dx$ 的定義，幾何意義和性質。



- Riemann sum = $\sum_{i=1}^{n} f(x_i^*)\Delta x, \quad \Delta x = \dfrac{b-a}{n}$

  $\overset{f(x)\geq 0}{=} \sum$ 小長方形面積

- Definite integral of f from a to

  $= \int_a^b f(x)dx := \lim_{n\to\infty}$ Riemann sum

  $= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \overset{f(x)\geq 0}{=}$ the area under the curve $y = f(x)$ from a to b.

  (一些無窮小項的無窮次累加)

- $f(x)$ is called the integrand; a is the lower limit; b is the upper limit.

Theorem：

If f is continuous on $[a, b]$.

$\Rightarrow \int_a^b f(x)dx$ is well-defined.

That is, the limit of the Riemann sum is independent of the choice of x_i^*

- Well-definedness of $\int_a^b f(x)dx$.

  (＊不等分的子區間也對)

- ∑：離散加法、∫：連續加法

Example 1：Evaluate the Riemann sum for $f(x)=x^2$ for $n=6$, x_i^* = the left endpoints, a = 0, b = 3.

Solution：

$$\frac{1}{2}\left[0^2+\left(\frac{1}{2}\right)^2+1^2+\left(\frac{3}{2}\right)^2+2^2+\left(\frac{5}{2}\right)^2\right]=\frac{55}{8}.$$

Example 2：Compute $\int_0^3\left(x^3-6x\right)dx$ by the definition.

Solution：

$$S_n=\sum_{i=1}^{n}\frac{3}{n}f\left(\frac{3i}{n}\right)=\sum_{i=1}^{n}\frac{3}{n}f\left(\frac{27i^3}{n^3}-\frac{18i}{n}\right)$$

$$=\frac{81}{n^4}\sum_{i=1}^{n}i^3-\frac{54}{n^2}\sum_{i=1}^{n}i=\frac{81n^2(n+1)^2}{4n^4}-\frac{27n(n+1)}{n^2}=\frac{81}{4}-27=\frac{-27}{4}.$$

Example 3：Evaluate $\int_0^1\sqrt{1-x^2}\,dx$ by interpreting in terms of area.

Solution：

$\int_0^1\sqrt{1-x^2}\,dx$=四分之一的單位圓面積=$\frac{1}{4}$.

Example 4：Find the " ?'s ", that is, identify f, a & b.

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+\left(\frac{i}{n}\right)^2}=\int_?^? ?\,dx$$

Solution：

$$a=0,\quad b=1,\quad f(x)=\frac{1}{1+x^2}.$$

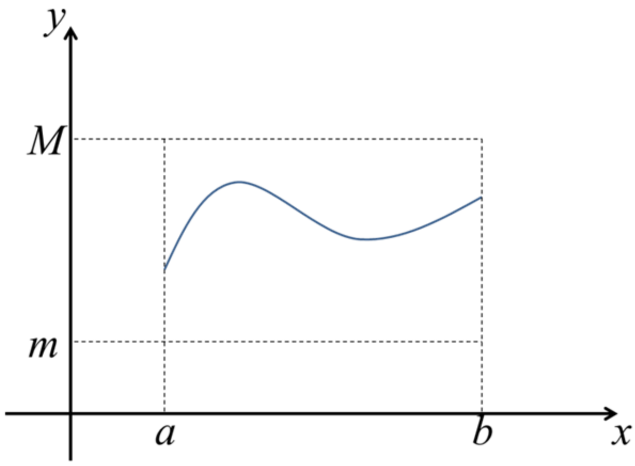
＊註解：這種問題解答不是唯一。

＊Properties of the integral

- $\int_a^b f(x)dx=-\int_b^a f(x)dx.$
- $\int_a^a f(x)dx=0$
- $\int_a^b(\alpha f(x)+\beta g(x))dx=\alpha\int_a^b f(x)dx+\beta\int_a^b g(x)dx$

  (微分、積分一樣都是線性運算)

- $\int_a^c f(x)dx+\int_c^b f(x)dx=\int_a^b f(x)dx$, for any $c\in R$.
- $f(x)\ge 0$ on $[a,b]\Rightarrow\int_a^b f(x)dx\ge 0$.
- $f(x)\ge g(x)$ on $[a,b]\Rightarrow\int_a^b f(x)dx\ge\int_a^b g(x)dx$.
- If $m\le f(x)\le M$ for $a\le x\le b$, then $m(b-a)\le\int_a^b f(x)dx\le M(b-a)$.


y
M
m
a
b
x