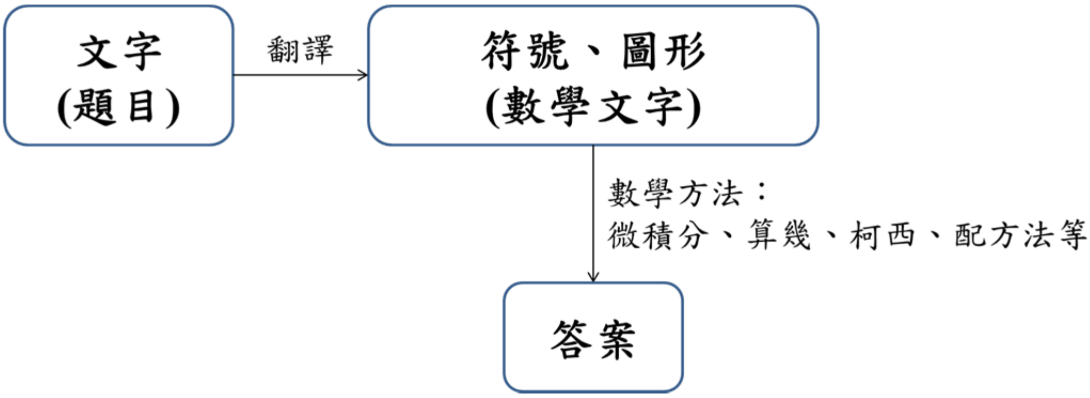
§4-7 Optimization



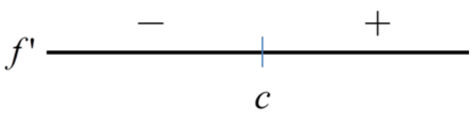
＊微積分方法：求最大值與最小值，可利用下列兩種方法。

1. f is continuous on $[\,a\,,\,b\,]$, 比較 critical points and endpoints.

x	a	critical points	b
y			

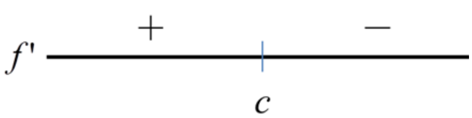
2. f has exactly one critical point c in the internal I.

If



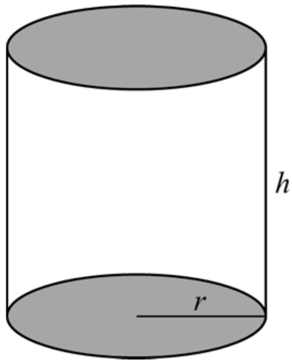
$\Rightarrow f\,(c)$ is an absolute minimum.

If



$\Rightarrow f\,(c)$ is an absolute maximum.

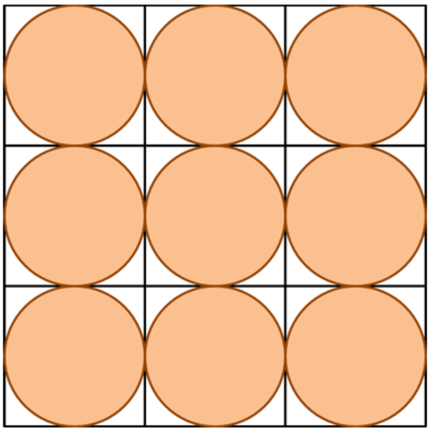
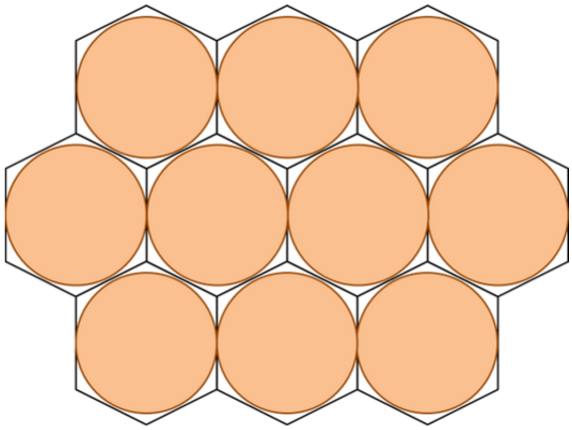
Example 1：A cylindrical can is to be made to hold 1l of oil.



(i) Find the dimensions that will minimize the cost the metal to manufacture the can? $(\frac{h}{r}=2)$

(ii) Suppose the top and bottom discs are cut from squares of side 2r. $(\frac{h}{r}=\frac{8}{\pi}\approx 2.55)$

(iii) How about discs cut from hexagons?

(ii)	(iii)
	

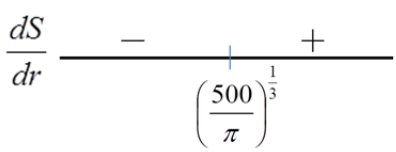
Solution：

Known that $V=\pi r^2 h=1000$

Minimize $A(r)=2\pi r^2+2\pi rh=2\pi r^2+\frac{2000}{r}, \quad 0<r<\infty.$

$$A'(r)=4\pi r-\frac{2000}{r^2}=\frac{4(\pi r^3-500)}{r^2}$$

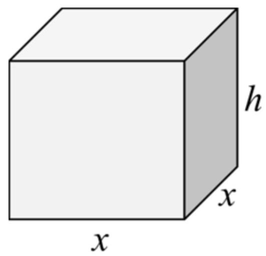
$$\Rightarrow r=\left(\frac{500}{\pi}\right)^{\frac{1}{3}}(cm)$$



$$\Rightarrow S\left[\left(\frac{500}{\pi}\right)^{\frac{1}{3}}\right] \text{ is an absolute minimum.}$$

$$\Rightarrow h = \frac{1000}{\pi\left(\frac{500}{\pi}\right)^{\frac{2}{3}}} = 2\sqrt[3]{\frac{500}{\pi}} = 2r\,.$$

Example 2： A box with a square base and open top must have a volume of 32000 cm^3. Find the dimensions of the box that minimize the amount of material of material used.

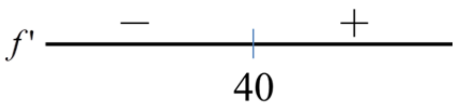


Solution：

Known that $V = x^2 h = 32000$

Minimize： $x^2 + 4xh = x^2 + \dfrac{128000}{x} \overset{\Delta}{=} f(x)$

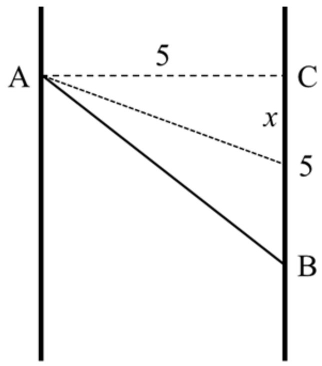
$$f'(x) = 2x - \frac{128000}{x^2} = \frac{2(x^3 - 64000)}{x^2}, \quad 0 < x < \infty .$$



$\Rightarrow x = 40,\ h = 20.$

國立交通大學應用數學系 莊重教授

Example 3：A man launches his boat from point A on a bank of a straight river, 5*km* wide, and wants to reach point B, 5*km* down on the opposite bank, as quick as possible. If he can row 6 *km*/*h* and run 8 *km*/*h*, where should he land to reach B as soon as possible?



Solution：

Let T be the time the man spend arriving B.

Minimize $T(x)=\frac{\sqrt{25+x^2}}{6}+\frac{5-x}{8},\quad 0\le x\le 5$

$\Rightarrow T'(x)=\frac{1}{6}\left(\frac{1}{2}\times\frac{2x}{\sqrt{25+x^2}}\right)-\frac{1}{8}$

Let $T'(x)=0$

$\Rightarrow \frac{1}{6}\left(\frac{x}{\sqrt{25+x^2}}\right)-\frac{1}{8}=0$

$\Rightarrow 4x=3\sqrt{25+x^2}$

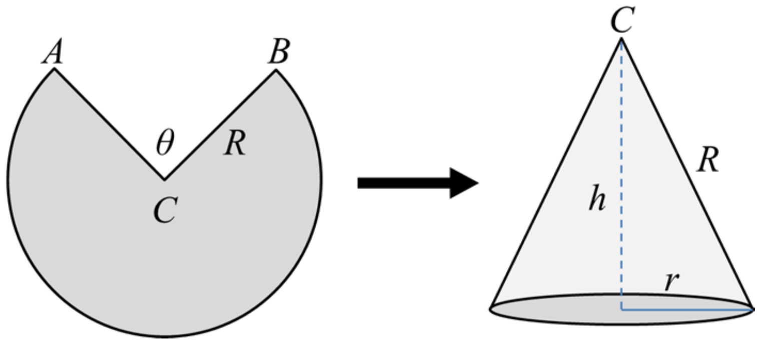
$\Rightarrow 16x^2=225+9x^2$

$\Rightarrow x=\frac{15}{\sqrt{7}}>5.$

x	0	5
y	$\frac{35}{24}\approx 1.458$	$\frac{5}{6}\sqrt{2}\approx 1.179$

He should row directly to B.

Example 4：A cone-shaped drinking cup is made from radius R by cutting out a sector and joining the edges CA and CB. Find the maximum capacity of such a cup.



Solution：

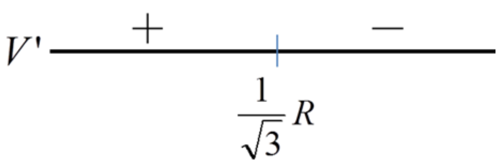
$$V(h) = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(R^2 - h^2)h$$

$$= \frac{\pi}{3}R^2 h - \frac{\pi}{3}h^3, \quad 0 < h < R.$$

$$\Rightarrow V'(h) = \frac{\pi}{3}(R^2 - 3h^2)$$

Let $V'(h) = 0$

$$\Rightarrow h = \frac{1}{\sqrt{3}}R$$

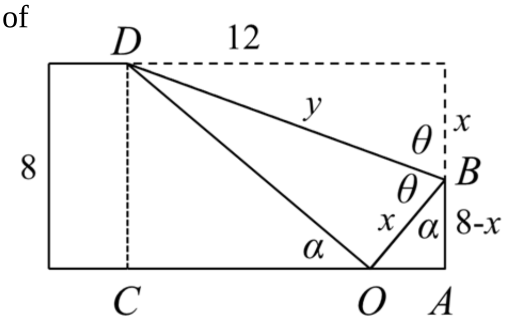


$$\Rightarrow V(\frac{1}{\sqrt{3}}R) = \frac{2\pi}{9\sqrt{3}}R^3$$ is the maximum.

＊想一想：為達最大容量，角度θ要取多少？

Ans：$\theta = 2\pi\left(1 - \frac{\sqrt{6}}{3}\right) \approx 66.06°$.

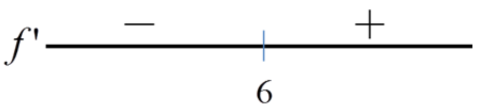
Example 5：The upper right-hand corner of a piece of paper, 12 *cm* by 8 *cm*, as in the figure, is folded over to the bottom edge. How would you fold it, so as to minimize the length of the fold ? In other words, how would you choose x to minimize y?



Solution：

$$\Delta OCD \sim \Delta BAO$$
$$\Rightarrow \frac{8}{\sqrt{y^2 - x^2}} = \frac{4\sqrt{x-4}}{x}$$
$$\Rightarrow \left(y^2 - x^2\right) \times (x-4) = 4x^2, \quad 4 < x \le 8 \quad (why?)$$
$$\Rightarrow y^2 = \frac{4x^2}{x-4} + x^2 = \frac{x^3}{x-4} \overset{\Delta}{=} f(x), \quad 4 < x \le 8.$$

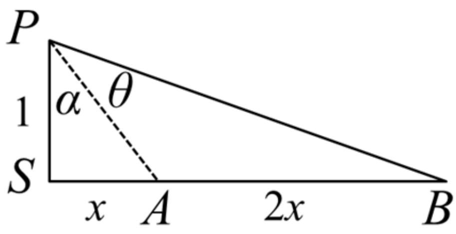
$$\Rightarrow f'(x) = \frac{2x^2(x-6)}{(x-4)^2} = 0$$
$$\Rightarrow x = 6.$$



$$\Rightarrow y = 6\sqrt{3}.$$

＊想一想：這個問題和紙張的長邊長度有關嗎？

Example 6：An observer stands at a point P, one unit away from a track. Two runners stand at the point S in the figure and run along the track. One runner runs three times as fast as other. Find the maximum value of the observer's angle of sight between runners?



Solution：

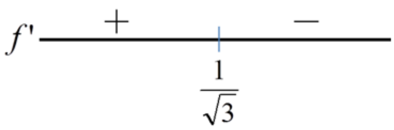
Runner B runs three times as fast as runner A

$$\tan\theta = \tan(\theta+\alpha-\alpha) = \frac{\tan(\theta+\alpha)-\tan\alpha}{1+\tan(\theta+\alpha)\tan\alpha}$$

$$= \frac{3x-x}{1+3x^2} = \frac{2x}{1+3x^2} \overset{\Delta}{=} f(x), \quad 0 \le x < \infty.$$

$$\Rightarrow f'(x) = \frac{2(1-3x^2)}{(1+3x^2)^2} = 0$$

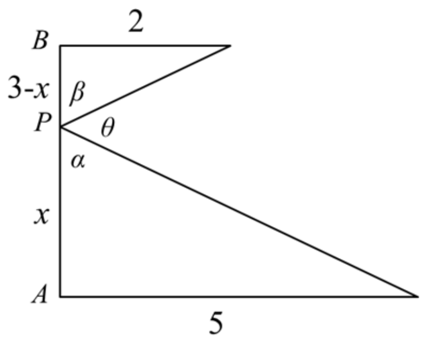
$$\Rightarrow x = \frac{1}{\sqrt{3}} \quad (x \ge 0)$$



$$\Rightarrow \tan\theta = \frac{2\frac{1}{\sqrt{3}}}{1+3\times\left(\frac{1}{\sqrt{3}}\right)^2} = \frac{1}{\sqrt{3}}$$

$\Rightarrow \theta = \frac{\pi}{6}$ is the max angle of sight.

Example 7： Where should the point P be chosen on the line segment AB so as to maximize the angle θ ?



Solution：

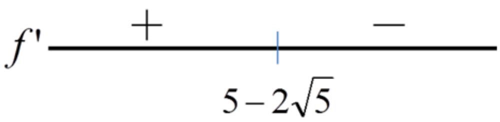
$$\tan\theta = \tan(\pi-(\alpha+\beta)) = -\tan(\alpha+\beta)$$

$$= -\left(\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\right) = \frac{\frac{5}{x}+\frac{2}{3-x}}{\left(\frac{5}{x}\times\frac{2}{3-x}\right)-1}$$

$$= \frac{3(5-x)}{x^2-3x+10} \overset{\Delta}{=} f(x), \quad 0\le x\le 5.$$

$$f'(x) = \frac{3(x^2-10x+5)}{(x^2-3x+10)^2} = 0$$

$$\Rightarrow x = 5\pm 2\sqrt{5} \quad (5+2\sqrt{5}>3\text{不合})$$



$\Rightarrow x = 5-2\sqrt{5}$ will maximize the angle θ.