§4-3 & §4-5 1st and 2nd Derivatives and Curve Sketching

＊ f', f''和 f 的圖形關係：

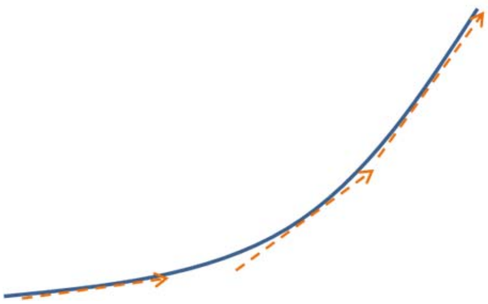
一、

1. $f'(x) > 0$ on (a , b) $\Rightarrow f$ is increasing on (a , b).

2. $f'(x) < 0$ on (a , b) $\Rightarrow f$ is decreasing on (a , b).

二、

1.



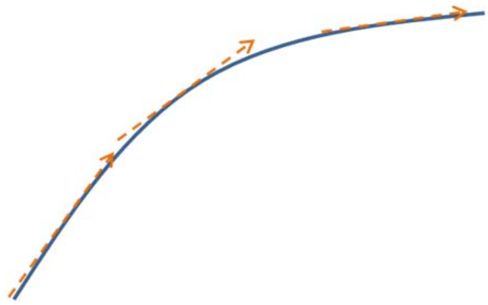
concave up (C.U.) on (a , b)

$\Leftrightarrow$ 圖形在切線上方.

f' is increasing on (a , b)

$\Leftrightarrow f''(x) > 0$ on (a , b).

2.



concave down (C.D.) on (a , b)

$\Leftrightarrow$ 圖形在切線下方.

f' is decreasing on (a , b)

$\Leftrightarrow f''(x) < 0$ on (a , b).

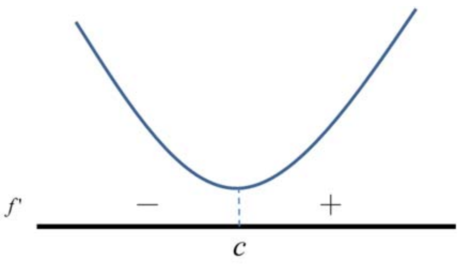
＊註記：

完整的證明可由 2 階 Taylor’s 展開式來證明。

三、Inflection point：the point at which concavity is reversed.(反曲點)

四、1^{st} Derivative Test：

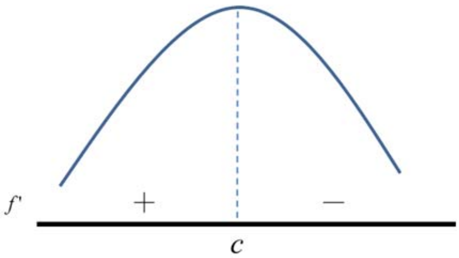
1.



$f' = 0 \Rightarrow f(c)$ is local an min.

＊c is a critical point.

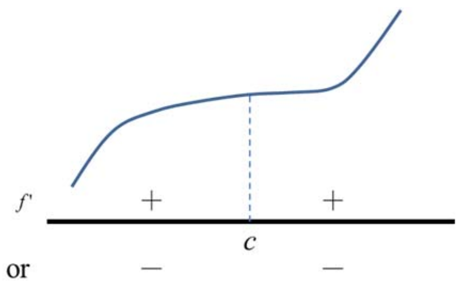
2.



$f' = 0 \Rightarrow f(c)$ is local an max.

＊c is a critical point.

3.



$f(c)$ is neither an min nor an max.

五、2^{nd} Derivative Test：

c：critical point

1. $f''(c) > 0 \Rightarrow f(c)$：local min (∵在 c 的附近，f is C.U.)

2. $f''(c) < 0 \Rightarrow f(c)$：local max (∵在 c 的附近，f is C.D.)

3. $f''(c) = 0 \Rightarrow$The test fails.

For example： $f(x) = x^3$, $f''(0) = 0 \Rightarrow f(0)$ is neither a min nor a max.

六、Curve Sketching：

1. Intercepts　　2. Symmetry　　3. Asymptotes
4. Local extrema　　5. Inflection points

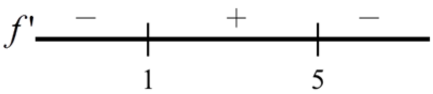
Example 1：Sketch the function $f(x) = 2 - 15x + 9x^2 - x^3$.

Solution：

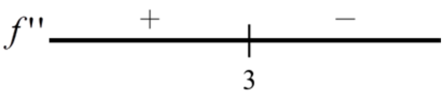
$$f(x) = 2 - 15x + 9x^2 - x^3 = -(x-2)(x^2 - 7x + 1)$$

$$f'(x) = -15 + 18x - 3x^2 = -3(x-5)(x-1)$$

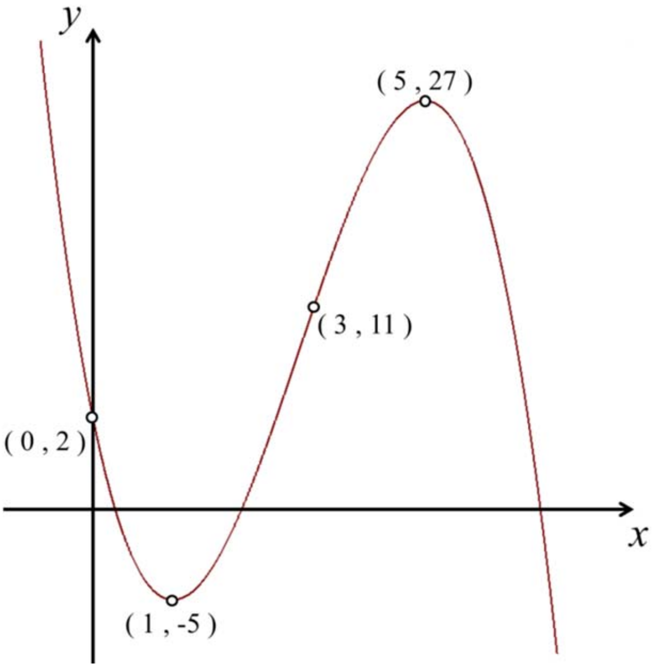
∗Critical points：x = 1 and 5.



$$f''(x) = 18 - 6x = -6(x-3)$$



x	0	1	2	3	5
y	2	-5	0	11	27



Example 2：Sketch the function $f(x)=x^{\frac{2}{3}}(6-x)^{\frac{1}{3}}$.

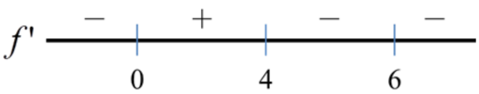
Solution：

$$f'(x)=\frac{2}{3}x^{-\frac{1}{3}}(6-x)^{\frac{1}{3}}-\frac{1}{3}x^{\frac{2}{3}}(6-x)^{-\frac{2}{3}}$$

$$=\frac{1}{3}x^{-\frac{1}{3}}(6-x)^{-\frac{2}{3}}(2(6-x)-x)$$

$$=x^{-\frac{1}{3}}(6-x)^{-\frac{2}{3}}(4-x)$$

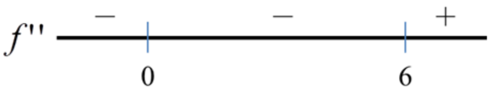
∗Critical points：$x = 0, 4$ and 6.



$$f''(x)=-\frac{1}{3}x^{-\frac{4}{3}}(6-x)^{-\frac{2}{3}}(4-x)+\frac{2}{3}x^{-\frac{1}{3}}(6-x)^{-\frac{5}{3}}(4-x)-x^{-\frac{1}{3}}(6-x)^{-\frac{2}{3}}$$

$$=\frac{1}{3}x^{-\frac{4}{3}}(6-x)^{-\frac{5}{3}}\left[-(6-x)(4-x)+2x(4-x)-3x(6-x)\right]$$

$$=\frac{-8}{x^{\frac{4}{3}}(6-x)^{\frac{5}{3}}}$$

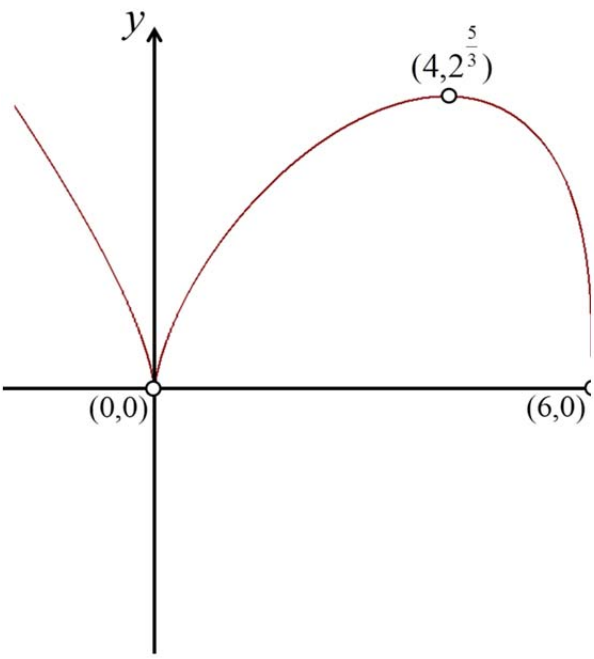


x	0	4	6
y	0	$2^{\frac{5}{3}}$	0

$\Rightarrow f(0) = 0$ is a local min.

$f(4) = 2^{\frac{5}{3}}$ is a local max.

$(6, 0)$ is an inflection point.

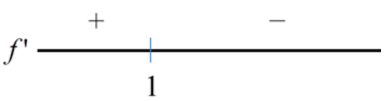


Example 3：Sketch the function $f(x) = xe^{-x}$.

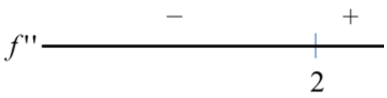
Solution：

$$f'(x) = e^{-x} - xe^{-x}$$
$$= e^{-x}(1-x)$$

∗Critical point：$x = 1$.



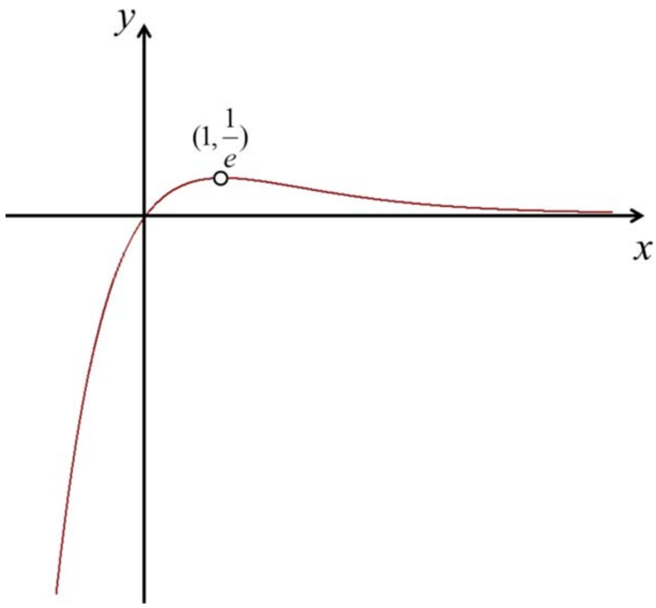
$$f''(x) = -e^{-x}(1-x) - e^{-x}$$
$$= -e^{-x}(2-x)$$



$\Rightarrow f(1) = \dfrac{1}{e}$ is local max.

$(2, \dfrac{2}{e^2})$ is an inflection point.

x	1	2
y	$\dfrac{1}{e}$	$\dfrac{2}{e^2}$



Example 4：Sketch the function $f(x) = \dfrac{x^3}{x^2+1}$

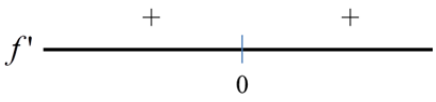
Solution：

$$y = f(x) = \frac{x^3}{x^2+1} = x - \frac{x}{x^2+1}$$

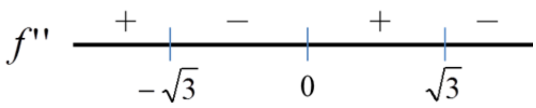
$\Rightarrow$Slant asymptotic：$y = x$.

$$f'(x) = \frac{x^2(x^2+3)}{(x^2+1)^2}$$

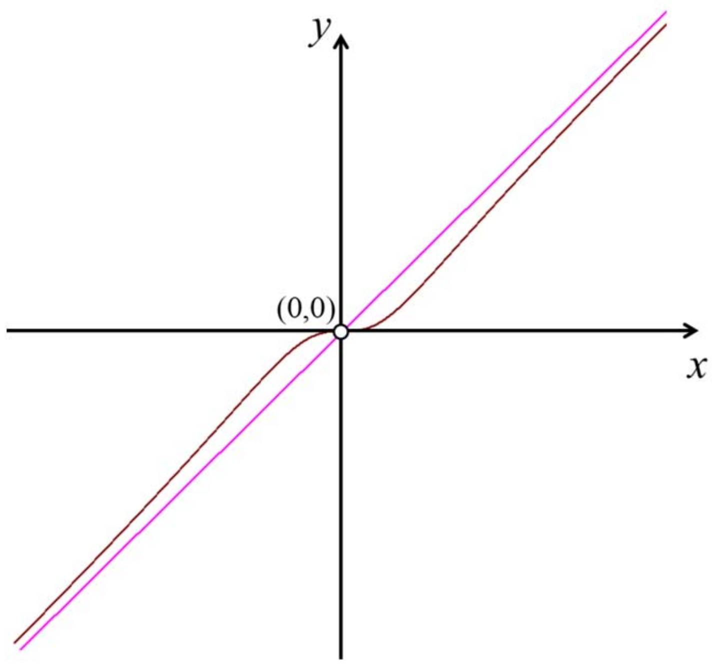
＊Critical point：$x = 0$.



$$f'(x) = \frac{-2x(x^2-3)(x^2+1)}{(x^2+1)^4}$$



x	$-\sqrt{3}$	0	$\sqrt{3}$
y	$\frac{3\sqrt{3}}{4}$	0	$-\frac{3\sqrt{3}}{4}$



Example 5：For what values of c is the function $f(x) = cx + \frac{1}{x^2+3}$ increasing on $(-\infty, \infty)$?

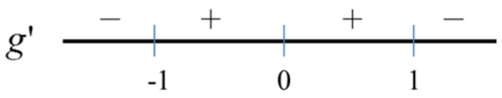
Solution：

$$f'(x) = c - \frac{2x}{(x^2+3)^2} = \frac{c(x^2+3)^2 - 2x}{(x^2+3)^2}$$

Hope：find $c > 0$ s.t. $c(x^2+3)^2 - 2x > 0 \Leftrightarrow c - \frac{2x}{(x^2+3)^2} > 0$

Let $g(x) = \frac{2x}{(x^2+3)^2} \Rightarrow g'(x) = \frac{-6(x^2-1)}{(x^2+3)^3}$

因為 g 對稱於原點，$\lim_{x\to\infty} g(x) = 0$，且



$\Rightarrow$ the max of g is $g(1) = \frac{1}{8}$.

$\Rightarrow$ If $c > \frac{1}{8}$, then $c(x^2+3)^2 - 2x > 0$ for all $x \in (-\infty, \infty)$.

$\Rightarrow f'(x) > 0$ for all $x \in (-\infty, \infty)$.

＊討論：1. What happens when $c = \frac{1}{8}$?

2. What happens when $c < \frac{1}{8}$?