§4-2 The Mean Value Theorem

∗Rolle's Theorem：

Assumption：

(1) f is continuous on [a , b]
(2) f is differentiable on (a , b)
} 簡稱 f is smooth on [a , b]

(3) $f(a) = f(b)$ (等高)

$\Rightarrow \exists\, c \in (a,b)$ s.t $f'(c) = 0$

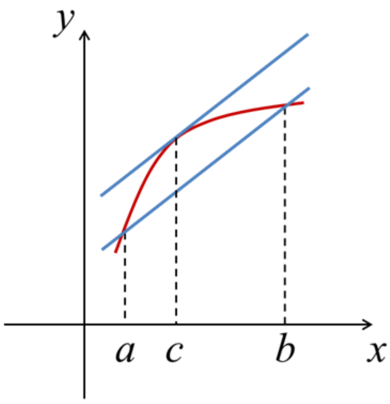
∗Smooth (平滑) + 等高：

$\Rightarrow \begin{cases} \text{1.數學: 有切線斜率為 0 的點(即有峰或谷)} \\ \text{2.物理: 有速度為 0 的時間} \end{cases}$

∗Mean Value Theorem (MVT)：

f is smooth on [a , b].

$\Rightarrow \exists\, c \in (a,b)$ s.t $f(b) - f(a) = f'(c)(b-a)$.



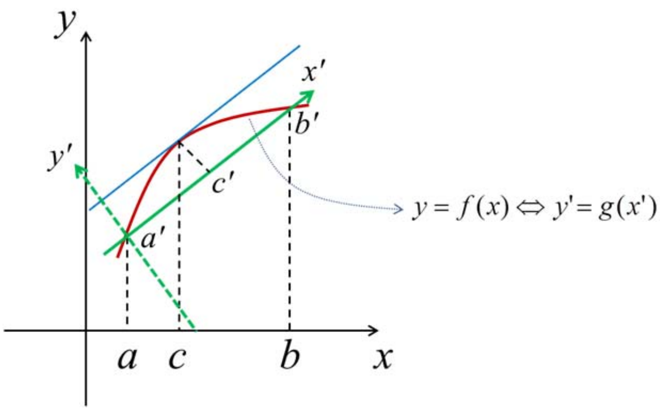
$\frac{f(b)-f(a)}{b-a} =$ 割線斜率

$f'(c) =$ 切線斜率

∗若 f 在 [a , b] 是平滑的 $\Rightarrow \begin{cases} \text{數學上: 割線斜率=某一切線斜率} \\ \text{物理上: 平均速度=某一瞬間速度} \end{cases}$

國立交通大學應用數學系 莊重教授

Proof of MVT：



Proof：在 x－y 新座標系，將 Rolle's Theorem 用在 $y' = g(x')$ 上.

$\Rightarrow$(a , $f(a)$)、(b , $f(b)$)都在新的 x 軸上.

$\Rightarrow \exists\, c' \in (0, b')$ $s.t$ $\quad g'(c) \,//\, x'$ 軸

$\Leftrightarrow \exists\, c \in (a, b)$ $s.t$ $\quad f'(c) \,//\, x'$ 軸

$$\Leftrightarrow f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Example 1： $f(x) = x^3 - x, \quad [a, b] = [0, 2]$, find c.

Solution：

$$\frac{6-0}{2-0} = 3c^2 - 1 \Rightarrow c = \frac{2\sqrt{3}}{3}.$$

Example 2：Let f be smooth, $\; f(0) = -3,\; f'(x) \le 5,\; 0 \le x \le 2.$

How large can $f(2)$ possible be ?

Solution：

$$\frac{f(2) - (-3)}{2 - 0} = f'(c) \le 5 \Rightarrow f(2) \le 7 \Rightarrow$$ The largest value can $f(2)$ be is 7.

Example 3：Let f be smooth, $f(0)=-3,\ f'(x)\geq 2,\ 0\leq x\leq 4.$

How small can $f(4)$ possible be ?

Solution：

$$\frac{f(4)-(-3)}{4-0}=f'(c)\geq 2\Rightarrow f(4)\geq 5\Rightarrow$$ The smallest value can $f(4)$ be is 5.

Example 4：Does there exist a function f s.t.

$f(0)=-1,\ f(2)=4$, and $f'(x)\leq 2$ for all x ?

Solution：

$$\frac{4+1}{2-0}=\frac{5}{2}>f'(c)\Rightarrow \text{impossible.}$$

Example 5：Prove that $\sqrt{1+x}<1+\frac{1}{2}x$ for $x>0$.

Proof：

Let $f(x)=\sqrt{1+x}\Rightarrow f'(x)=\frac{1}{2}(1+x)^{-\frac{1}{2}}$

$$\overset{MVT}{\Rightarrow} f(x)-f(0)=f'(c)(x-0),\ c\in(0,x)$$
$$\Rightarrow \sqrt{1+x}-1=\frac{1}{2}(1+c)^{-\frac{1}{2}}(x)$$
$$\Rightarrow \sqrt{1+x}=1+\frac{1}{2}(1+c)^{-\frac{1}{2}}(x)<1+\frac{1}{2}x.$$

(Since $(1+c)^{-\frac{1}{2}}<1$)

Example 6：Prove that $|\cos a - \cos b| \le |a-b|$ for all *a* and *b*.

Proof：

Let $f(x)=\cos x$, assume without loss of generality, that $a \ge b$.

$\overset{MVT}{\Rightarrow} \cos a - \cos b = f'(c)(a-b) = (\sin c)(a-b)$

$\Rightarrow |\cos a - \cos b| \le |a-b|$.

(Since $|\sin c| < 1$)

Example 7：(Theorem)

If $f'(x)=0$ on (*a* , *b*) ⇒ *f* is constant on (*a* , *b*).

Proof：

Let $x_1, x_2 \in (a,b), x_1 < x_2$

$\Rightarrow f(x_1) - f(x_2) = f'(c)(x_1 - x_2) = 0, \quad c \in (x_1, x_2)$

$\Rightarrow f(x_1) = f(x_2)$

⇒ *f* has the same value at any two numbers $x_1, x_2 \in (a,b)$

⇒ *f* is constant on (*a* , *b*).

Example 8：Let $f'(x) = g'(x)$ for all $x \in (a,b)$

$\Rightarrow f(x) = g(x) + c$ for all $x \in (a,b)$.(Here *c* is a constant)

Proof：

A direct consequence of **Example 7.**