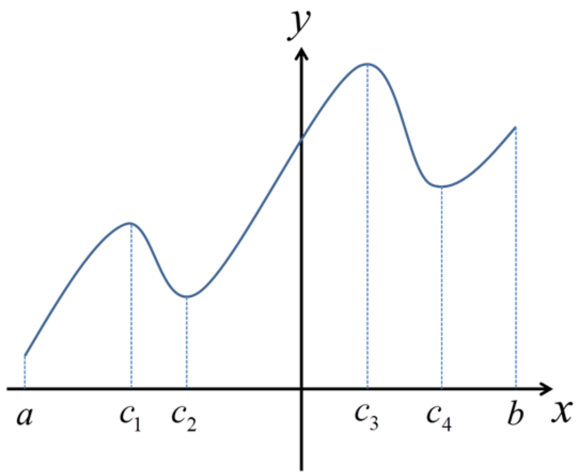
§4-1 Maximum and Minimum Values

＊ Absolute extreme (Global extreme)

＊ Relative extreme (Local extreme)

＊討論：$f(c\ ) > f(c\ )$



f：$[a,b] \to \boldsymbol{R}$

Absolute maximum： $f(c_3)$ (i.e. f has an absolute maximum at $x = c_3$.)

Absolute minimum： $f(a)$.

Relative maximum： $f(c_1), f(c_3), f(b)$.

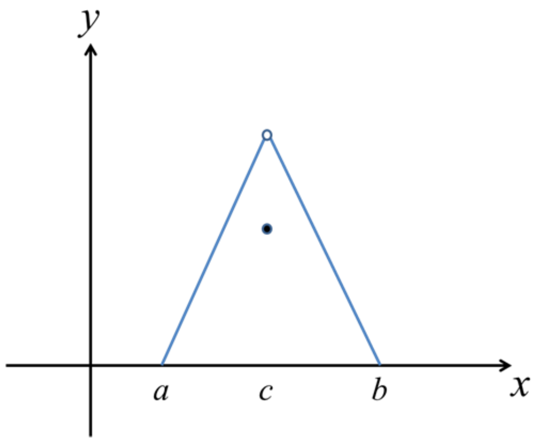
Relative minimum： $f(a), f(c_2), f(c_4)$.

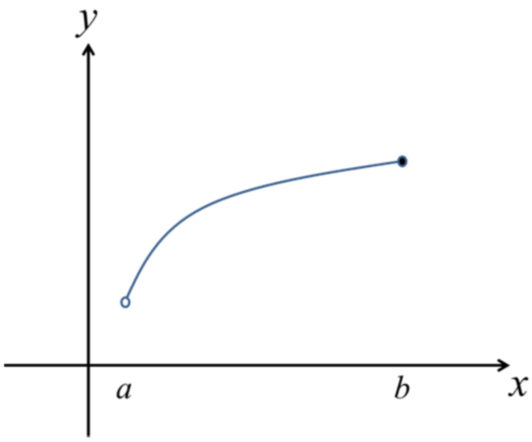
＊Note that $f(c_4) > f(c_1)$.

Theorem：

f is continuous on $[\,a\,,b\,] \rightarrow f$ attains it's absolute extreme.

＊註記：
Assumptions of the theorem need to be satisfied.
Otherwise the assertion of the theorem fails.



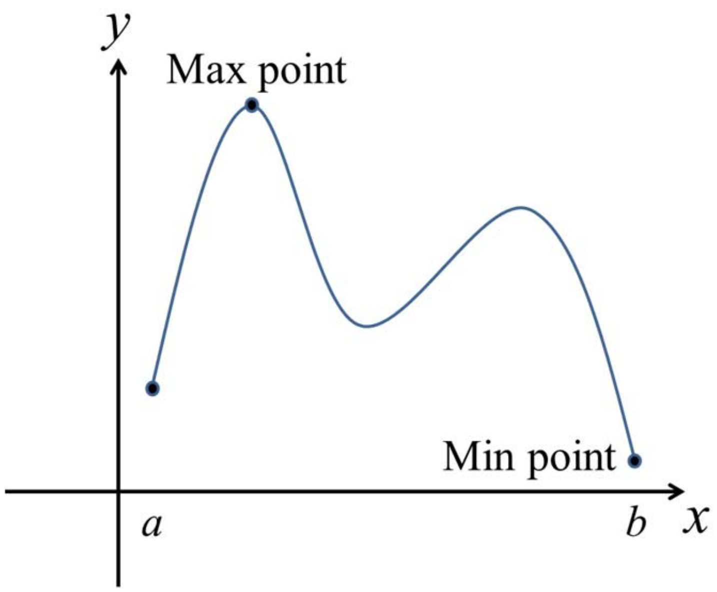


＊ Not continuous at $x = c$.

＊ No absolute maximum.

＊Not continuous at $x = a$.

＊No absolute minimum.



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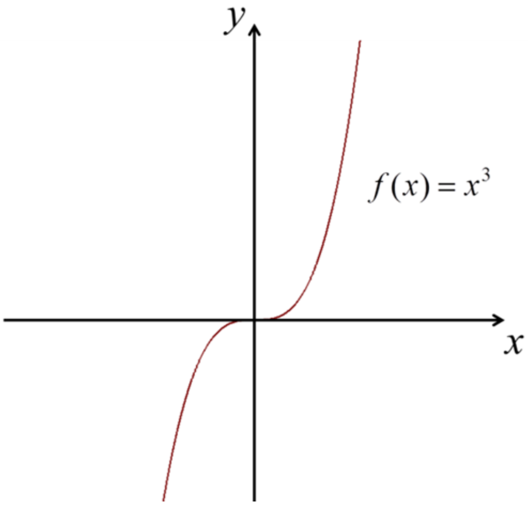
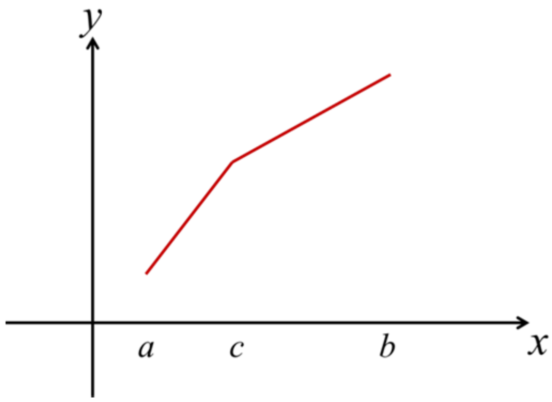
Theorem：

f has a local extremum at $x = c \rightarrow \ f'(c) = 0$ or $f'(c)$ does not exist.

＊註記：
The converse of the theorem is not necessary true.



$f(x) = x^3$, $f'(0) = 0$, but $f(0)$ is neither a local min nor a local max.



$f'(c)$ does not exist.

$f(c)$ is neither a local min nor a local max.

Definition：

If c is a critical point (c.p.) of $f \Leftrightarrow \ f'(c) = 0$ or $f'(c)$ does not exist.

Remark：

Extrema take place at critical points.

＊註記：
A critical point is not necessarily a local extremum point.

＊Steps to find absolute extrema for a continuous function defined on $[a, b]$.

1. Find all c.p.'s.
2. Compare the values of f at all c.p.'s and endpoints.

Example 1：Find the extrema of $f(x) = x^3 + 3x^2 + 1, \quad -\frac{1}{2} \le x \le 4.$

Solution：

$$f'(x) = 3x^2 + 6x = 3x(x+2)$$

$\Rightarrow$ Critical points：$x = 0.$

x	$-\frac{1}{2}$	0	4
$f(x)$	$\frac{13}{8}$	1	113

$$\Rightarrow \begin{cases} \max : 113 \\ \min : 1. \end{cases}$$

Example 2：Let $a, b > 0$, find the extrema of $f(x) = x^a (1-x)^b, \quad 0 \le x \le 1.$

Solution：

$$\begin{aligned} f'(x) &= a x^{a-1}(1-x)^b - b x^a (1-x)^{b-1} \\ &= x^{a-1}(1-x)^{b-1}\left[a(1-x) - bx\right] \\ &= x^{a-1}(1-x)^{b-1}\left[a - (a+b)x\right] \end{aligned}$$

$\Rightarrow$ Critical points： $x = 0, \frac{a}{a+b}, 1.$

x	0	$\frac{a}{a+b}$	1
$f(x)$	0	$\left(\frac{a}{a+b}\right)^a \left(\frac{b}{a+b}\right)^b$	0

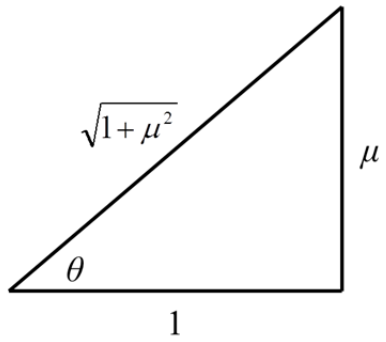
$$\Rightarrow \begin{cases} \max : \left(\frac{a}{a+b}\right)^a \left(\frac{b}{a+b}\right)^b \\ \min : 0. \end{cases}$$

Example 3：Find the extrema of $F(\theta)=\dfrac{\mu\omega}{\mu\sin\theta+\cos\theta}$, $0\le\theta\le\dfrac{\pi}{2}, \omega>0.$

Solution：

$$F'(\theta)=\frac{-\mu\omega(\mu\cos\theta-\sin\theta)}{(\mu\sin\theta+\cos\theta)^2}=0$$

$$\Rightarrow \theta=\tan^{-1}\mu$$



θ	0	$\dfrac{\pi}{2}$	$\tan^{-1}\mu$
$F(\theta)$	$\mu\omega$	ω	$\dfrac{\mu\omega}{\sqrt{1+\mu^2}}$

$\Rightarrow F$ is minimized when $\tan\theta=\mu$.

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