# §3-9 Related Rates

＊Related Rates：討論某些量 $x$ 對時間 $t$ 的變化率(rate of change)的應用問題。

$\left.\begin{array}{l} \textit{At what rate......x.....} \\ \textit{How fast ...........x.....} \end{array}\right\}$ means that $\frac{dx}{dt} = ?$(即 $x$ 對時間 $t$ 的變化率).

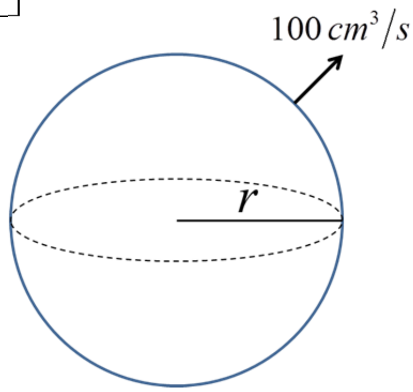
Steps to solve related rates problems：

1. 引進"符號"，將題目(文字)轉換成"圖"。
2. 找一方程式"連結"此圖。
3. 將方程式對時間 $t$ 微分。($\frac{d}{dt}$(此方程式))

**Example 1**： Air is being pumped into a spherical balloon so that its volume increases at a rate of $100\ cm^3/s$. How fast is the radius of the balloon increasing when the diameter is $50\ cm$?

**Solution**：

Known that $\frac{dV}{dt} = 100(cm^3/s)$, find $\left.\frac{dr}{dt}\right|_{r=25} = ?$




$$V = \frac{4}{3}\pi r^3$$

$$\Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{25}{\pi r^2}$$

$$\Rightarrow \left.\frac{dr}{dt}\right|_{r=25} = \frac{1}{25\pi}\left(cm/s\right).$$

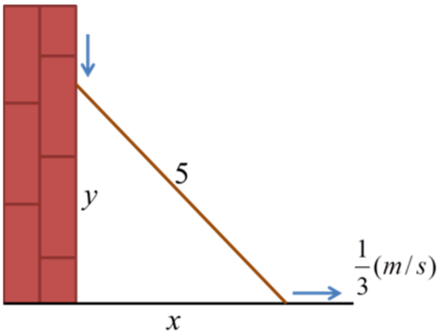


國立交通大學應用數學系 莊重教授

**Example 2**：A ladder 5 $m$ long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of $\frac{1}{3} m/s$. How fast is the top of the ladder sliding down the wall when the ladder is 3 $m$ from the wall?

**Solution**：

| |
|---|
| Known that $\frac{dx}{dt} = \frac{1}{3}(m/s)$, find $\left.\frac{dy}{dt}\right\|_{x=3} = ?$ |




$$x^2 + y^2 = 25$$
$$\Rightarrow 2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right) = 0$$

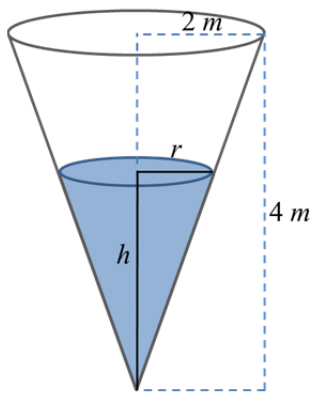
And known that when $x = 3$, $y = 4$.

$$\left.\frac{dy}{dt}\right|_{x=3} = -\frac{1}{4}(m/s).$$

**Example 3**：A water tank has the shape of an inverted circular cone with base radius 2 $m$ and height 4 $m$. If water is being pumped into the tank at a rate 2 $m^3/\text{min}$, find the rate at which the water level is rising when the water is 3 $m$ deep.

**Solution**：

| |
|---|
| Known that $\frac{dV}{dt} = 2(m^3/\text{min})$, find $\left.\frac{dh}{dt}\right\|_{h=3} = ?$ |



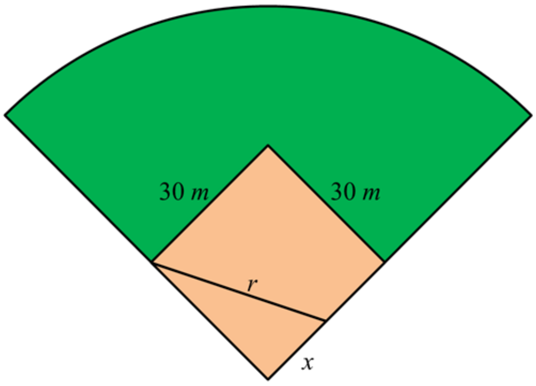

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3$$
$$\Rightarrow \frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$
$$\Rightarrow \left.\frac{dh}{dt}\right|_{h=3} = \frac{8}{9\pi}(m/\text{min}).$$

**Example 4**：A baseball diamond is a square with 30 *m*. A batter hits the ball and runs toward first base with a speed of 8 *m/s*. At what rate is his distance from the third base increasing when he is half way to first base?

**Solution**：

Known that $\frac{dx}{dt} = 8(m/s)$,

find $\left.\frac{dr}{dt}\right|_{x=15} = ?$




$$30^2 + x^2 = r^2$$

$$\Rightarrow 2x\frac{dx}{dt} = 2r\frac{dr}{dt}$$

And known that when $x = 15$, $r = 15\sqrt{5}$

$$\Rightarrow \left.\frac{dr}{dt}\right|_{x=15} = \frac{8\sqrt{5}}{5}(m/s).$$

**Example 5**：The minute hand on a watch is 8 *mm* long and the hour hand is 4 *mm* long. How fast is the distance between the tips of the hands changing at one o'clock?

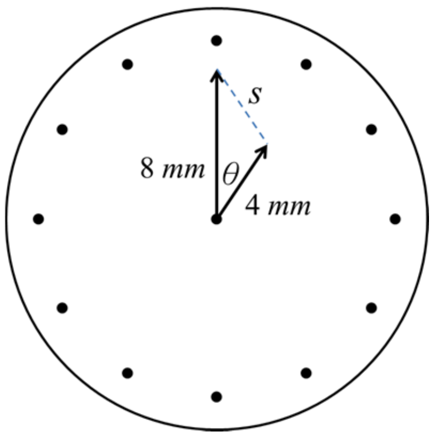
**Solution**：

Known that the angular velocity：

minute hand： $2\pi(rad/h)$

hour hand： $\frac{\pi}{6}(rad/h)$

$$\Rightarrow \frac{d\theta}{dt} = \frac{\pi}{6} - 2\pi = -\frac{11}{6}\pi(rad/h),$$



find $\left.\frac{ds}{dt}\right|_{\theta=\frac{\pi}{6}} = ?$

$$s^2 = 64 + 16 - 64\cos\theta$$

$$\Rightarrow 2s\frac{ds}{dt} = 64\sin\theta\frac{d\theta}{dt}$$

and known that when $\theta = \frac{\pi}{6}$, $s = \sqrt{80 - 32\sqrt{3}}$

$$\left.\frac{ds}{dt}\right|_{\theta=\frac{\pi}{6}} = \frac{-88\pi}{3\sqrt{80-32\sqrt{3}}}(mm/h) \approx -18.59(mm/h)\,.$$