# §3-6 Derivatives of Logarithmic Functions

＊公式

(I). $\frac{d}{dx}\left(\log_a x\right)=\frac{1}{x\ln a}$.

$\frac{d}{dx}\ln x=\frac{1}{x}$.

$\frac{d}{dx}\ln|x|=\frac{1}{x}$.

(II). $(x^r)'=rx^{r-1}, \quad r\in R$.

(III). $e=\lim_{x\to 0}(1+x)^{\frac{1}{x}}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

$e^x=\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n, \quad x>0.$

Proof of (I)：

$$
\begin{aligned}
& y=\log_a x \\
& \Rightarrow x=a^y \\
& \Rightarrow 1=a^y\times\left(\ln a\right)\times y' \\
& \Rightarrow y'=\frac{1}{a^y\ln a}=\frac{1}{x\ln a}.
\end{aligned}
$$

if $x<0, \ln|x|=\ln(-x)\Rightarrow\frac{d}{dx}\ln|x|=\frac{d}{dx}\ln(-x)=\frac{-1}{-x}=\frac{1}{x}$,

if $x>0, \ln|x|=\ln x\Rightarrow\frac{d}{dx}\ln|x|=\frac{d}{dx}\ln x=\frac{1}{x}$.

Proof of (II)：

$$y = x^r \Rightarrow \ln|y| = r\ln|x|$$
$$\Rightarrow \frac{y'}{y} = \frac{r}{x}$$
$$\Rightarrow y' = r\frac{y}{x} = rx^{r-1}.$$

Proof of (III)：

Let $f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1.$

$$1 = f'(1) = \lim_{x\to 0}\frac{f(1+x) - f(1)}{x}$$
$$= \lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}\frac{1}{x}\ln(1+x)$$
$$= \lim_{x\to 0}\ln(1+x)^{\frac{1}{x}} = \ln\left(\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right)$$
$$\Rightarrow \lim_{x\to 0}(1+x)^{\frac{1}{x}} = e.$$

＊註記：最後一個等式用到連續函數和極限可交換的性質。

**Example 1**：Let $y = \log(x^3+1)$, find $y' = ?$

**Solution**：

$$y' = \frac{3x^2}{(\ln 10)(x^3+1)}.$$

**Example 2**：Let $y = \frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$, find $y' = ?$

**Solution**：

$$\ln y = \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$
$$\Rightarrow \frac{y'}{y} = \frac{3}{4x} + \frac{2x}{2(x^2+1)} - \frac{15}{3x+2}$$
$$\Rightarrow y' = \left(\frac{3}{4x} + \frac{2x}{2(x^2+1)} - \frac{15}{3x+2}\right)y.$$

**Example 3**：Let $y = x^x$, find $y' = ?$

(Hint：If $y = x^2$, then $y' = 2x$. If $y = 2^x$, then $y' = 2^x \ln 2$.)

**Solution**：

$$\ln y = x \ln x$$
$$\Rightarrow \frac{y'}{y} = \ln x + \frac{x}{x}$$
$$\Rightarrow y' = x^x(\ln x + 1).$$

**Example 4**：Let $y = x^{\cos x}$, find $y' = ?$

**Solution**：

$$\ln y = \cos x \ln x$$
$$\Rightarrow \frac{y'}{y} = -\sin x \times \ln x + \frac{\cos x}{x}$$
$$\Rightarrow y' = x^{\cos x}(-\sin x \times \ln x + \frac{\cos x}{x}).$$

**Example 5**：Evaluate $\frac{d^9}{dx^9}(x^8 \ln x)$.

**Solution**：

$$\frac{d^9}{dx^9}(x^8 \ln x)$$
$$= \frac{d^8}{dx^8}(8x^7 \ln x + \frac{x^8}{x}) = D^8(8x^7 \ln x)$$
$$= D^7(8 \times 7x^6 \ln x + 8x^6) = D^7(8 \times 7x^6 \ln x)$$
$$= D(8! \times \ln x) = \frac{8!}{x}.$$