§3-5 Implicit Differentiation

＊**Question**：一個二元方程式是否隱藏函數的關係？

Example：

(i). $x+y=1$ 能否寫成

$x+f(x)=1$, where $y=f(x)$?

(ii). $x^2+y^2=1$ 能否寫成

$x^2+\left(f(x)\right)^2=1$, where $y=f(x)$?

(iii). $x^3+y^3=6xy$ 能否寫成

$x^3+\left(f(x)\right)^3=6x\left(f(x)\right)$, where $y=f(x)$?

(iv). 給一個二元方程式 $g(x,y)=0$，能否將此方程式寫成

$g(x,f(x))=0$?

＊註記：
很明顯(i)的答案是肯定的，(ii)、(iii)、(iv)的答案是較複雜的。數學系大二高微課程中的隱函數定理(Implicit Function Theorem)會給出較完整的答案。粗略的講，除了少數的點之外，其他點的局部附近，上述問題的答案都是肯定的。

在初微的課程中，我們都假設：
The given equation determines y (respectively, x) implicitly as a differentiable function of x (respectively, y) so that the method of implicit differentiation can be applied.

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Example 1：

Let $x^2+y^2=25$, find $\frac{dy}{dx}$, $\frac{dx}{dy}$ and the equation of the tangent at (3 , 4).

Solution：

$$2x+2y\left(\frac{dy}{dx}\right)=0 \qquad\qquad 2x\left(\frac{dx}{dy}\right)+2y=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{x}{y}. \qquad\qquad \Rightarrow \frac{dx}{dy}=-\frac{y}{x}.$$

$$m=\left.\frac{dy}{dx}\right|_{(3,4)}=-\frac{3}{4}\Rightarrow (y-4)=-\frac{3}{4}(x-3).$$

Example 2：Let $x^3+y^3=6xy$, find $\frac{dy}{dx}=?$

Solution：

$$3x^2+3y^2\left(\frac{dy}{dx}\right)=6y+6x\left(\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{2y-x^2}{y^2-2x}.$$

Example 3：Let $\sin(x+y)=y^2\cos x$, find $\frac{dy}{dx}=?$

Solution：

$$\cos(x+y)\times\left(1+\frac{dy}{dx}\right)=2y\left(\frac{dy}{dx}\right)\cos x-y^2\sin x$$

$$\Rightarrow \frac{dy}{dx}=\frac{\cos(x+y)+y^2\sin x}{2y\cos x-\cos(x+y)}.$$

Example 4：Let $f(f^{-1}(x)) = x$.

(i). Find $\left(f^{-1}\right)'(x) = ?$

(ii). If $f(4) = 5$, $f'(4) = \frac{2}{3}$, find $\left(f^{-1}\right)'(5) = ?$

Solution：

(i). $\left(f(f^{-1}(x)) = x\right)' \Rightarrow f'(f^{-1}(x)) \times \left(f^{-1}(x)\right)' = 1$

$$\Rightarrow \left(f^{-1}\right)'(x) = \frac{1}{f'(f^{-1}(x))}.$$

(ii). $\left(f^{-1}\right)'(5) = \frac{3}{2}$.

Example 5：The **Bessel function** of order 0, $y = J(x)$, satisfies the differential equation $xy'' + y' + xy = 0$ for all values of x and its value at 0 is $J(0) = 1$.

(i). Find $J'(0) = ?$

(ii). Use implicit differentiation to find $J''(0) = ?$

Solution：

(i). 將 $x = 0$ 代入微分方程，得 $J'(0) = y'(0) = 0$.

(ii). 將微分方程兩邊對 x 微分得

$$y'' + xy''' + y'' + y + xy' = 0$$

$$\Rightarrow y''(0) = J''(0) = -\frac{1}{2}.$$

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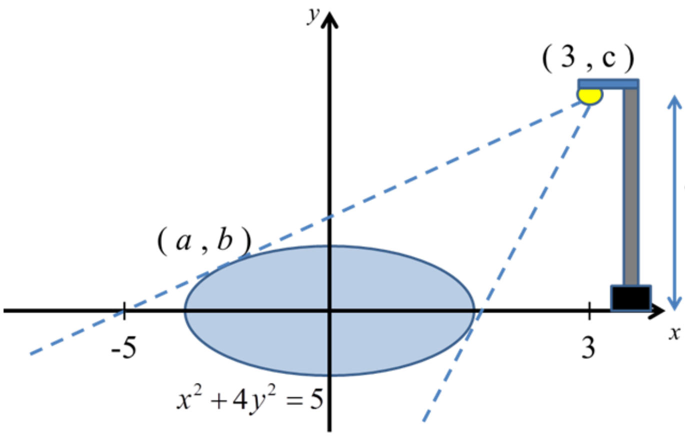
Example 6：The figure shows a lamp located three units to the right of the y-axis and a shadow created by the elliptical region $x^2+4y^2 \le 5$. If the point $(-5,0)$ is on the edge of the shadow, how far above the x-axis is the lamp located?



Solution：

Let (a , b) be a point on the ellipse.

將 $x^2+4y^2=5$ 對 x 微分，

$$\Rightarrow 2x+8yy'=0 \Rightarrow y'=-\frac{x}{4y}=-\frac{a}{4b}$$ (在橢圓上過點(a , b)的斜率)

$$\Rightarrow \begin{cases} a^2+4b^2=5 \\ \dfrac{b}{a+5}=-\dfrac{a}{4b} \end{cases} \Rightarrow \begin{cases} a^2+4b^2=5 \\ a^2+4b^2=-5a \end{cases} \Rightarrow \begin{cases} a=-1 \\ b=1 \end{cases}.$$

$$\Rightarrow \frac{1}{4}=\frac{c}{8} \Rightarrow c=2.$$

＊反三角函數的微分：

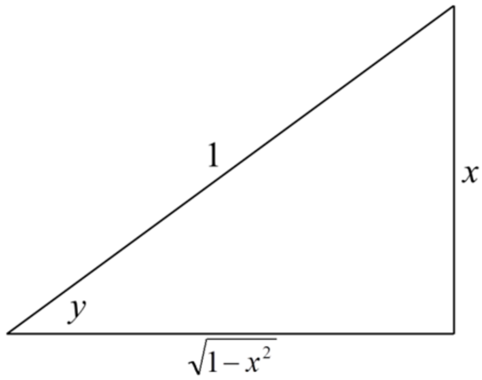
i. $(\sin^{-1}x)'=\dfrac{1}{\sqrt{1-x^2}}$ ii. $(\cos^{-1}x)'=\dfrac{-1}{\sqrt{1-x^2}}$

iii. $(\tan^{-1}x)'=\dfrac{1}{1+x^2}$ iv. $(\cot^{-1}x)'=\dfrac{-1}{1+x^2}$

v. $(\sec^{-1} x)' = \dfrac{1}{x\sqrt{x^2-1}}$ vi. $(\csc^{-1} x)' = \dfrac{-1}{x\sqrt{x^2-1}}$

($\csc^{-1} x$, $\sec^{-1} x \in$ third quadrant when $x < 0$)

Proof of (i)：



$$y = \sin^{-1} x$$

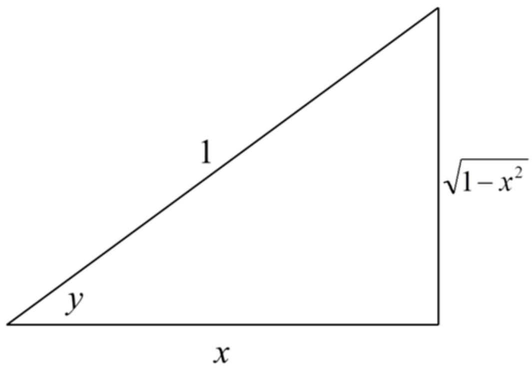
$$\Rightarrow x = \sin y$$

$$\Rightarrow 1 = (\cos y) \times \left(\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}.$$

Proof of (ii)：

＊(Method 1)：



$$y = \cos^{-1} x$$

$$\Rightarrow x = \cos y$$

$$\Rightarrow 1 = (-\sin y) \times \left(\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{-\sin y} = -\frac{1}{\sqrt{1-x^2}}$$

＊(Method 2)：

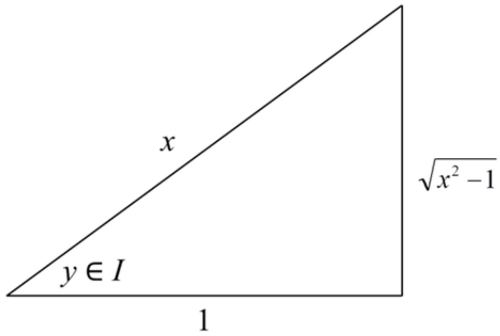
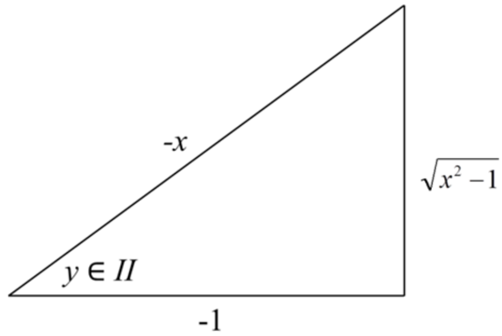
$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

$$\Rightarrow (\sin^{-1} x)' + (\cos^{-1} x)' = 0$$

$$\Rightarrow (\cos^{-1} x)' = -(\sin^{-1} x)' = \frac{-1}{\sqrt{1-x^2}}.$$

Proof of (v)：

$$y=\sec^{-1}x$$
$$\Rightarrow x=\sec y$$
$$\Rightarrow 1=\sec y\times\tan y\times\left(\frac{dy}{dx}\right)$$
$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sec y\times\tan y}=\frac{1}{x\sqrt{x^2-1}}$$

	
(i) $x>0,\ \ 0\le y<\frac{\pi}{2}$.	(ii) $x<0,\ \ \pi\le y<\frac{3\pi}{2}$.

Note that if $\sec^{-1}x\in$ second quadrant when $x<0$,

then $\frac{dy}{dx}=\begin{cases}\frac{1}{x\sqrt{x^2-1}} & x>0\\ \frac{1}{-x\sqrt{x^2-1}} & x<0.\end{cases}$

That is why in this book, $\sec^{-1}x$ is chosen to be in the third quadrant whenever $x<0$.