§3-3 Derivatives of Trigonometric Functions

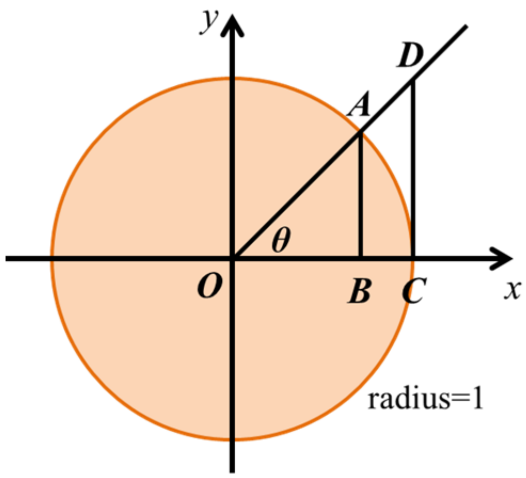
＊欲得三角函數的微分，需先得知(I)式中的極限值。

I.

i. $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1$

ii. $\lim\limits_{\theta\to 0}\dfrac{\cos\theta-1}{\theta}=0$

Proof of (I-i)：



$\overline{AB}<\widehat{AC}<\overline{CD}$

$\Downarrow \quad \Downarrow \quad \Downarrow$

$\Rightarrow \sin\theta<\theta<\tan\theta$.

$\Rightarrow 1<\dfrac{\theta}{\sin\theta}<\dfrac{\tan\theta}{\sin\theta}=\dfrac{1}{\cos\theta}$.

$\Rightarrow \lim\limits_{\theta\to 0^+}\dfrac{\theta}{\sin\theta}=1$.

Similarly, $\lim\limits_{\theta\to 0^-}\dfrac{\theta}{\sin\theta}=1 \Rightarrow \lim\limits_{\theta\to 0}\dfrac{\theta}{\sin\theta}=1$.

＊註記：

i. Why $\widehat{AC}(=\theta)<\overline{CD}$？
扇形 OAC 的面積<三角形 OCD 的面積
$\Rightarrow \dfrac{1}{2}\theta<\dfrac{1}{2}\widehat{AC}<0$.
$\Rightarrow \widehat{AC}<\overline{CD}$.

ii. 利用(I-i)和半角公式可得(I-ii)。

(也就是說，當 θ 很小的時候，$\sin\theta$ 差不多就是 θ！)

II.

i. $(\sin x)' = \cos x$　　ii. $(\cos x)' = -\sin x$

iii. $(\tan x)' = \sec^2 x$　　iv. $(\cot x)' = -\csc^2 x$

v. $(\sec x)' = \sec x \tan x$　　vi. $(\csc x)' = -\csc x \cot x$

Proof of (II-i)：

$$(\sin x)' = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$
$$= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}$$
$$= \sin x \lim_{h\to 0}\frac{\cos(h)-1}{h} + \cos x \lim_{h\to 0}\frac{\sin(h)}{h}$$
$$= \cos x$$

＊註記：

嚴格的寫法，需先說明第三等式兩個極限存在，因此第二等式等於第三等式。

Proof of (II-iv)：

$$(\cot x)' = \left(\frac{\cos x}{\sin x}\right)' = \left(\frac{\sin x(-\sin x)-\cos x(\cos x)}{\sin^2 x}\right) = \frac{-1}{\sin^2 x} = -\csc^2 x$$

Example 1：$\lim_{x\to 0}\frac{\sin 3x}{x} = ?$

Solution：

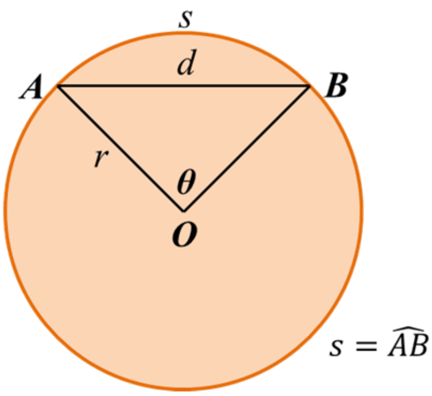
$$\lim_{x\to 0}\frac{\sin 3x}{x} = 3\lim_{x\to 0}\frac{\sin 3x}{3x} = 3.$$

Example 2：$\lim_{t\to 0}\frac{\sin^2 3t}{t^2} = ?$

Solution：

$$\lim_{t\to 0}\frac{\sin^2 3t}{t^2} = \lim_{t\to 0}\frac{(3t)^2}{t^2} = 9.$$

Example 3：



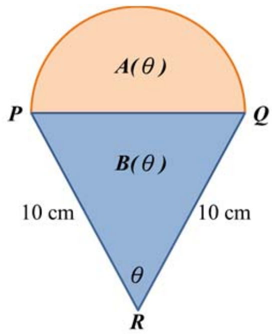
求 $\lim_{\theta \to 0^+} \frac{s}{d} = ?$

Solution：

$$\lim_{\theta \to 0^+} \frac{s}{d} = \lim_{\theta \to 0^+} \frac{r\theta}{2r\sin\frac{\theta}{2}} = \lim_{\theta \to 0^+} \frac{\theta}{2\sin\frac{\theta}{2}} = \lim_{\theta \to 0^+} \frac{\theta}{2\times\frac{\theta}{2}} = 1$$

Example 4： A semicircle with diameter *PQ* sits on an isosceles triangle *PQR* to form a region shaped like a two-dimensional ice cream cone, as shown in the figure. If $A(\theta)$ is the area of the semicircle and $B(\theta)$ is the area of the triangle, find $\lim_{\theta \to 0^+} \frac{A(\theta)}{B(\theta)}$.

Solution：



Known that the radius of the semicircle $= 10\sin\frac{\theta}{2}$.

$$\Rightarrow A(\theta) = \frac{1}{2}\times\pi(10\sin\frac{\theta}{2})^2 = 50\pi\sin^2(\frac{\theta}{2})$$

and $B(\theta) = \frac{1}{2}\times 10^2\sin\theta = 50\sin\theta$

$$\Rightarrow \lim_{\theta \to 0^+} \frac{A(\theta)}{B(\theta)} = \lim_{\theta \to 0^+} \frac{50\pi\sin^2(\frac{\theta}{2})}{50\sin\theta} = \pi\times\lim_{\theta \to 0^+} \frac{(\frac{\theta}{2})^2}{\theta} = 0.$$