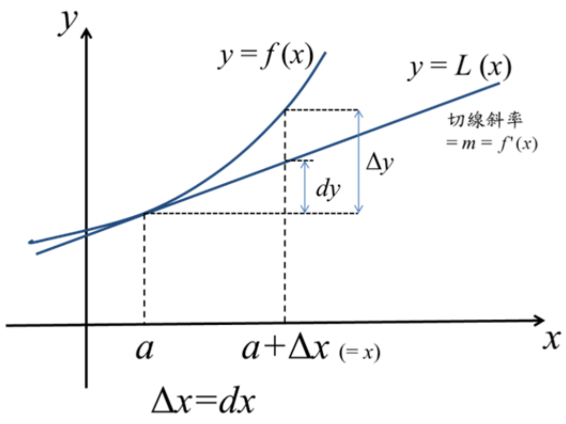
§3-10 Linear Approximations and Differentials

＊Given $f(x)$，利用“過 a 點的切線方程式”來逼近當 x 靠近 a 的 $f(x)$。



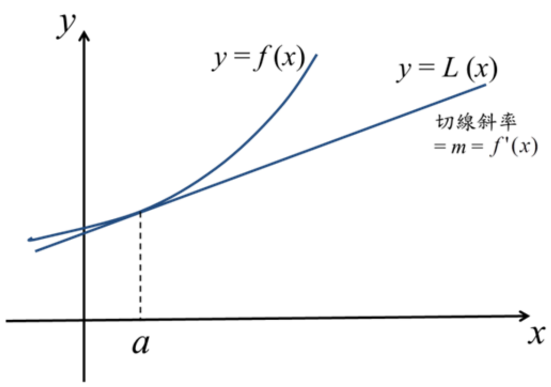
- $\Delta x = dx =$ initial error.
- Approximate error $= dy = f'(a)dx$.
- Exact error $= \Delta y = f(a+\Delta x) - f(a)$.
- dx and dy are called differentials.

＊公式：

1. $\Delta y \approx dy = f'(a)dx$ when $\Delta x = dx$ is small.

   在實際問題上，我們常能以 dy (較好算)來逼近 Δy (較難算)。

2. $L(x) = y = f(a) + f'(a)(x-a) = f(a) + dy$ is the linear approximation or tangent line approximation of f at a.



- $y = L(x)$是過(a , $f(a)$)和 $f(x)$相切的切線方程式。
- $\lim_{x\to a}(f(x) - L(x)) = 0$.

  (⇒誤差$= |f(x) - L(x)| \to 0$, as $x \to 0$.)

- $\lim_{x\to a}\dfrac{f(x) - L(x)}{x-a} = 0$

  (⇒誤差$\to 0$ 的速度比 $x - a \to 0$ 當 $x \to a$ 的速度還快).

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Example 1：

(i). Find the linearization of $f(x)=\sqrt{x+3}$ at $a=1$.

(ii). Use (i) to approximate $\sqrt{3.98}$ and $\sqrt{4.05}$.

Solution：

$$f'(x)=\frac{1}{2}(x+3)^{-\frac{1}{2}} \Rightarrow f'(1)=\frac{1}{4}$$

$$\Rightarrow L(x)=f(1)+f'(1)(x-1)=2+\frac{1}{4}(x-1).$$

$$\sqrt{3.98}\approx 2+\frac{1}{4}(0.98-1)=1.995.$$

$$\sqrt{4.05}\approx 2+\frac{1}{4}(1.05-1)=2.0125.$$

	x	From $L(x)$	Actual value
$\sqrt{3.9}$	0.9	1.975	1.97484176…
$\sqrt{3.98}$	0.98	1.995	1.99499373…
$\sqrt{4}$	1	2	2.00000000…
$\sqrt{4.05}$	1.05	2.0125	2.01246117…
$\sqrt{4.1}$	1.1	2.025	2.02484567…
$\sqrt{5}$	2	2.225	2.23606797…
$\sqrt{6}$	3	2.5	2.44948974…

Example 2： The radius of a sphere was measured and found to be 21 *cm* with a possible error in measurement of at most 0.05 *cm*. What is the maximum error in using the value of radius to compute the volume of the sphere? (Use differential to estimate such maximum error.)

Solution：

$$V=\frac{4}{3}\pi r^3$$

$$\Rightarrow \frac{dV}{dr}=4\pi r^2$$

$$\Rightarrow dV=4\pi r^2 dr=4\pi(21)^2(0.05)\approx 277\ cm^3.$$

Example 3：Approximate $\ln(1.05)$.

Solution：

Let $y = \ln x$

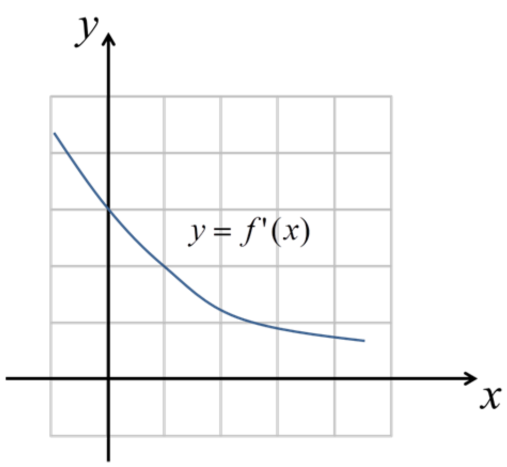
$$\Rightarrow dy = \frac{dx}{x} = \frac{0.05}{1} = 0.05$$

$$\Rightarrow \ln(1.05) \approx \ln 1 + 0.05 = 0.05.$$

Example 4：Suppose that the only information we have about a function f is that $f(1) = 5$ and the graph of its derivative is as shown.

(a) Use a linear approximation to estimate $f(0.9)$ and $f(1.1)$.

(b) Are your estimates in (a) too large or too small? Explain.



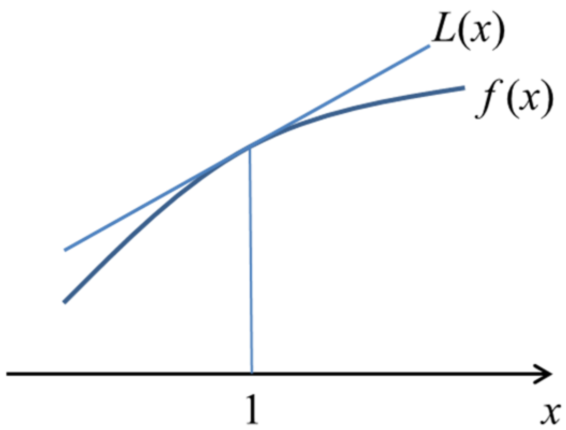
Solution：

Let $L(x)$ be the tangent line of $f(x)$ at $x = 1$.

$$\Rightarrow L(x) = f(1) + f'(1)(x-1) = 5 + 2(x-1)$$

$$\Rightarrow \begin{cases} f(0.9) \approx 5 + 2(-0.1) = 4.8 \\ f(1.1) \approx 5 + 2(0.1) = 5.2. \end{cases}$$

Since $f'(x) > 0$ and $f''(x) < 0$ whenever x is real, we have that the graph of $f(x)$ near $x = 1$ may looks like the following：



The tangent line lies above the curve. Thus, the approximations in part (a) are too large.