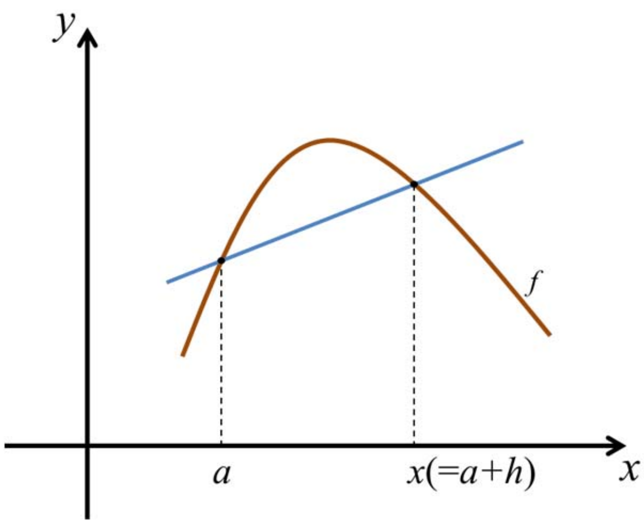
§2-7 Derivatives and Rates of Change

Definition：

f has a derivative at $x = a$ (or *f* is differentiable at $x = a$) provided that

$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ exists or equivalently $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ exists.

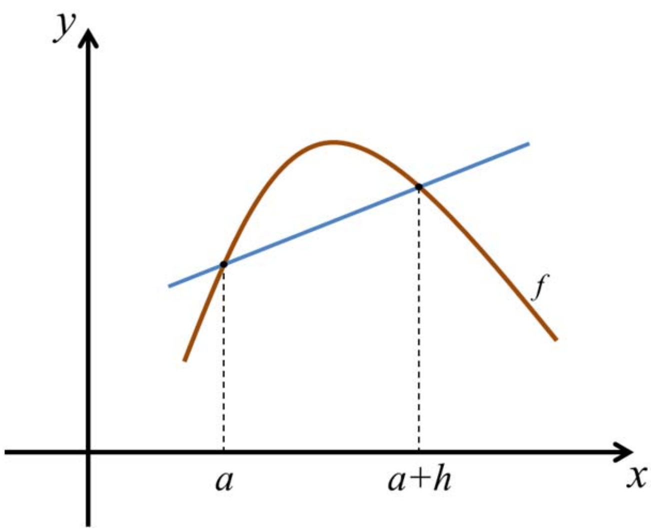
This limit will be denoted by $f'(a)$.



Remark：

Geometrically,

(i) $\frac{f(a+h)-f(a)}{h}$ = the slope of the secant.



國立交通大學應用數學系 莊重教授

(ii) $f'(a)$ = the slope of the tangent of $y = f(x)$ at $x = a$.

＊註記：Generally speaking： $f'(a)$ = **the rate of change** of f at a.

i. 若 $f(t)$ = position at time t.

$\Rightarrow f'(a)$ = the instantaneous velocity at time $t = a$.

ii. $f(x)$ = the cost to produce $x(m)$ amount of fabric.

$\Rightarrow f'(a)$ = the marginal cost when the production level is at $x = a$.

Example 1： $f(x) = x^2 - 8x + 9$. Compute $f'(a)$ by definition.

Solution：

$$\begin{aligned}
&\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\\
&=\lim_{x\to a}\frac{(x^2-8x+9)-(a^2-8a+9)}{x-a}\\
&=\lim_{x\to a}(x+a-8)\\
&=2a-8
\end{aligned}$$

Example 2：Find an equation of the tangent to $y = x^2 - 8x + 9$ at $(3,-6)$.

Solution：

Slope at $x = 3$： $f'(3) = -2$

Point： $(3,-6)$

Point-slope form (點斜式)： $y + 6 = -2(x - 3)$

Example 3：Let $c = f(v)$, where v denotes the speed $\left({km}/{hr} \right)$ of the car, and c means its gas consumption $\left({liter}/{hr} \right)$.

i. What is the meaning of the derivative $f'(v)$?

What is its unit?

ii. Write a sentence that can explains the meaning of the equation of $f'(40) = -0.1$.

Solution：

i. $f'(v)$ means the gas consumption rate if a car at speed v.

Its unit：$\dfrac{{liter}/{hr}}{{km}/{hr}} = {liter}/{km}$.

ii. The gas consumption is decreasing by $0.1 = {liter}/{km}$ as the speed of the car reaches $40\,{km}/{hr}$.

Example 4：The cost of producing x meters of a fabric is $C = f(x)$ dollars.

i. What is the meaning of the derivative $f'(x)$? What is its unit?
ii. In practical terms, what is the meaning of $f'(1000) = 9$?
iii. Which do you think is greater, $f'(50)$ or $f'(500)$? How about $f'(5000)$?

Solution：

i. The meaning of $f'(x)$ is the marginal cost when the production level is x. Unit= ${\$}/{m}$

ii. 當公司生產量是 1000m，如果額外再生產 1 公尺長度的布料，公司差不多只需額外花費 9 元。

iii. $f'(50)$：低產量

$f'(500)$：中產量

$f'(5000)$ ：高產量

$f'(50) > f'(500)$, $f'(5000) > f'(500)$.