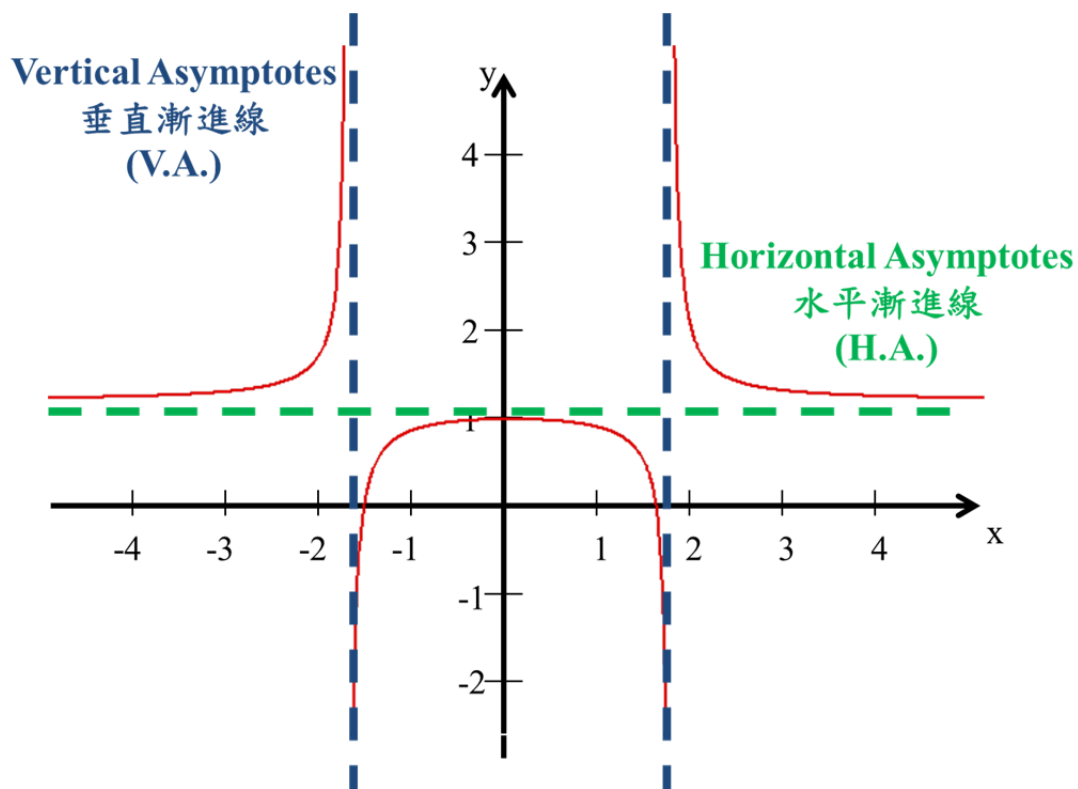


## §2-6 Limit at Infinity; Horizontal Asymptotes

- Vertical Asymptotes (垂直漸進線) (V.A.)

Horizontal Asymptotes (水平漸進線) (H.A.)



**Definition :**

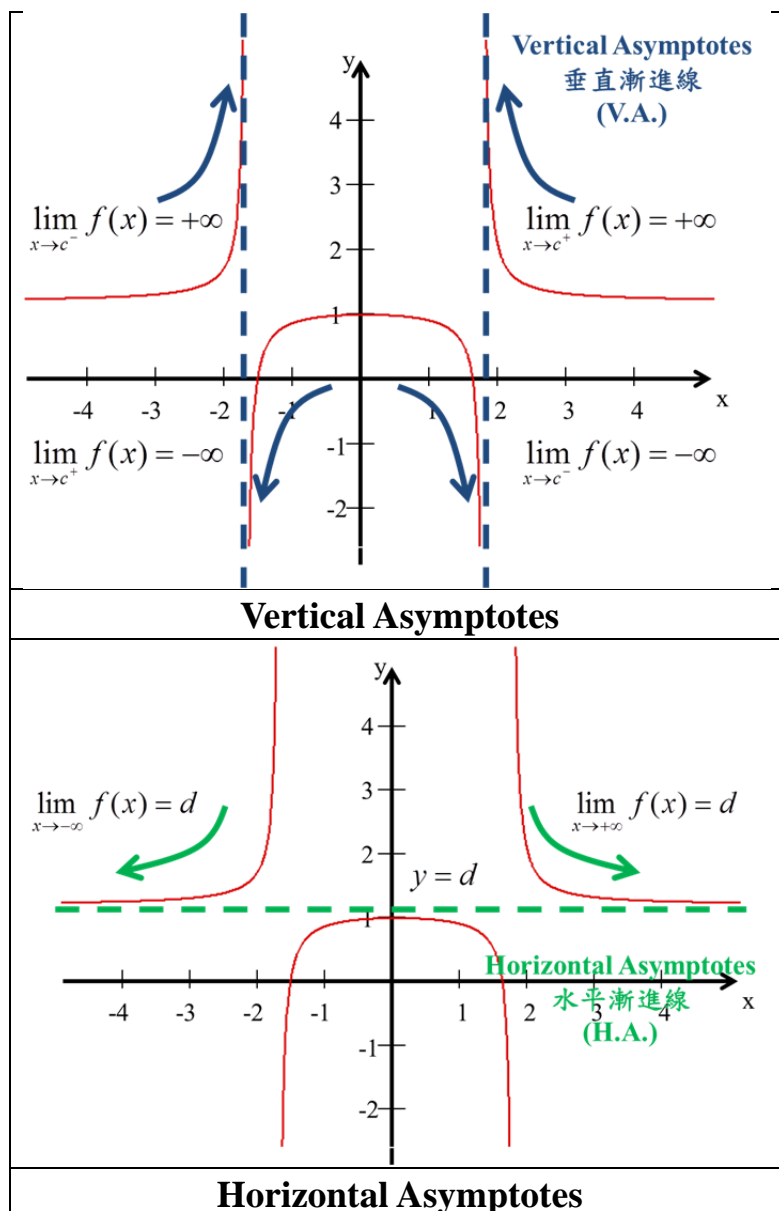
- (i) If either of the following 4 limits holds,

$$\lim_{x \rightarrow c^{\pm}} f(x) = \pm\infty,$$

then  $x = c$  is called a V.A. of  $f$ .

- (ii) If  $\lim_{x \rightarrow +\infty} f(x) = d$  or  $\lim_{x \rightarrow -\infty} f(x) = d$ ,

then  $y = d$  is called a H.A. of  $f$ .



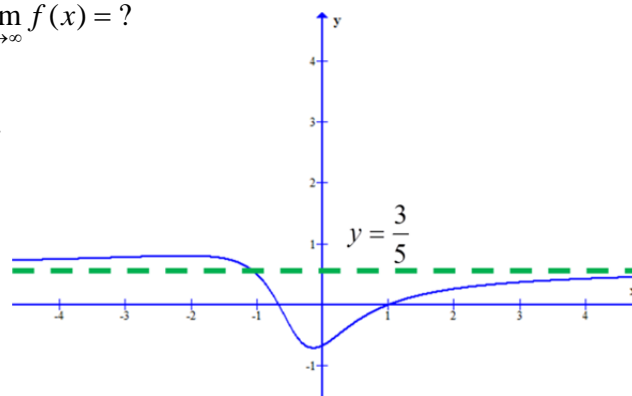
**Example 1 :** Let  $f(x) = \frac{3x^2 - x - 2}{5x^2 + 4x + 3}$ , find  $\lim_{x \rightarrow \infty} f(x) = ?$

And find H.A. & V.A. of  $f$  if it has.

**Solution :**

$$\lim_{x \rightarrow \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 3} = \lim_{x \rightarrow \infty} \frac{3x^2}{5x^2} = \frac{3}{5}$$

$$\text{H.A. : } y = \frac{3}{5}$$



V.A. : No.

( $\because 5x^2 + 4x + 3 > 0$  for all  $x \Leftrightarrow$  There is no  $c \in \mathbb{R}$  s.t  $\lim_{x \rightarrow c} (5x^2 + 4x + 3) = 0$ .)

**Example 2 :** Find H.A. & V.A. of the following  $f$ .

$$(i) f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}, (ii) f(x) = \frac{\sqrt{9x^6 - x}}{x^3 + 1}$$

**Solution :**

(i)

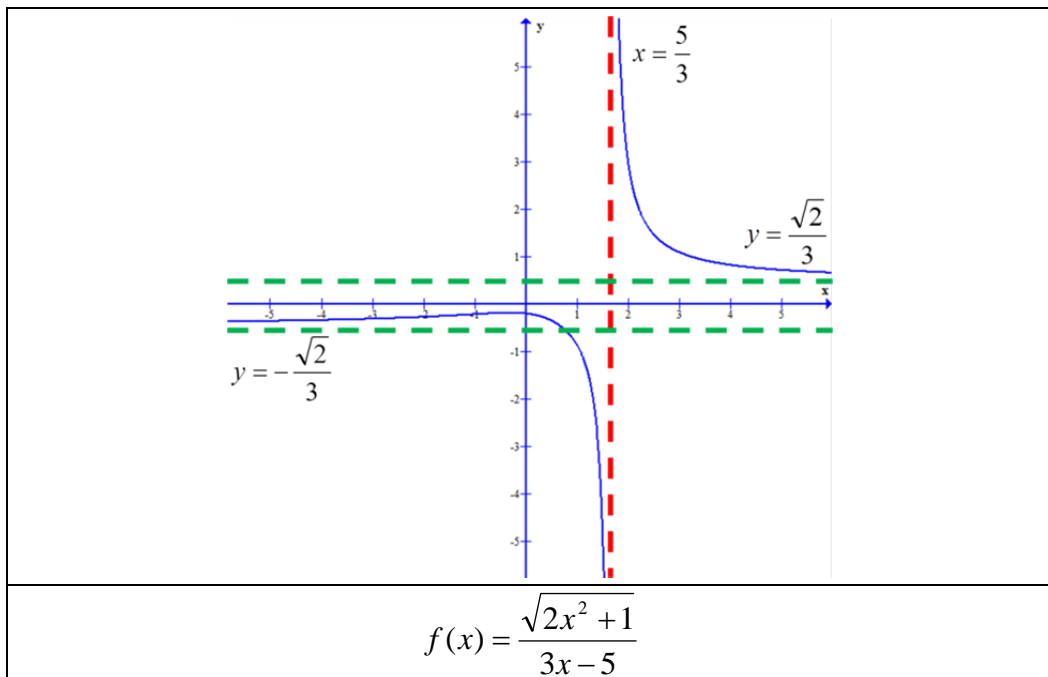
$$\text{H.A. : } y = \frac{\sqrt{2}}{3} \text{ or } y = -\frac{\sqrt{2}}{3}.$$

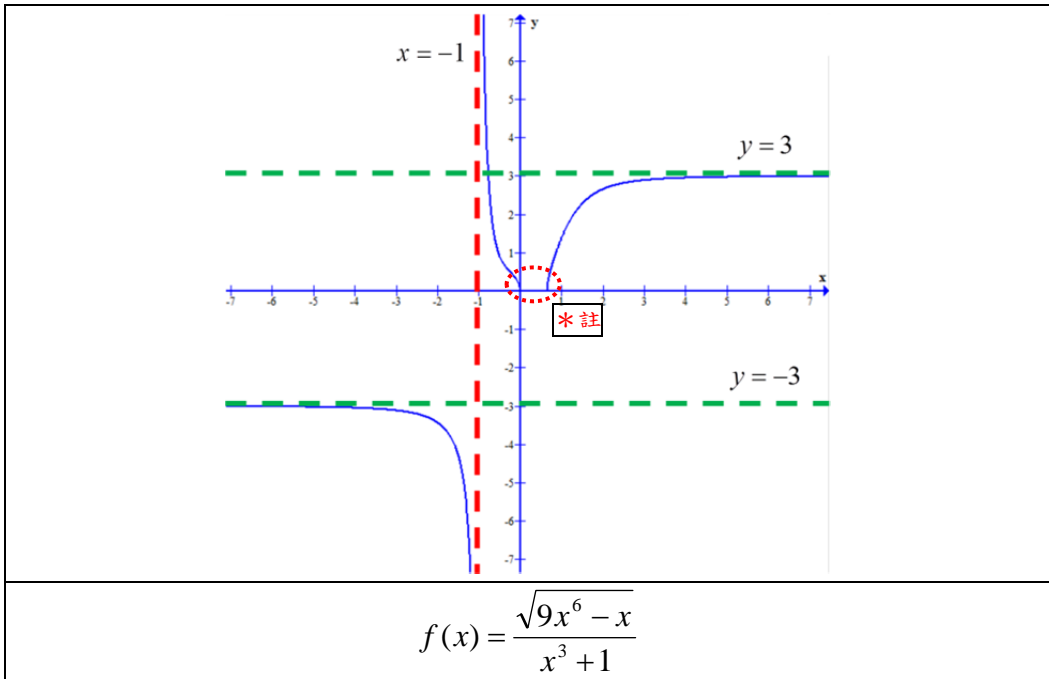
$$\text{V.A. : } x = \frac{5}{3}.$$

(ii)

$$\text{H.A. : } y = 3 \text{ or } y = -3.$$

$$\text{V.A. : } x = -1.$$



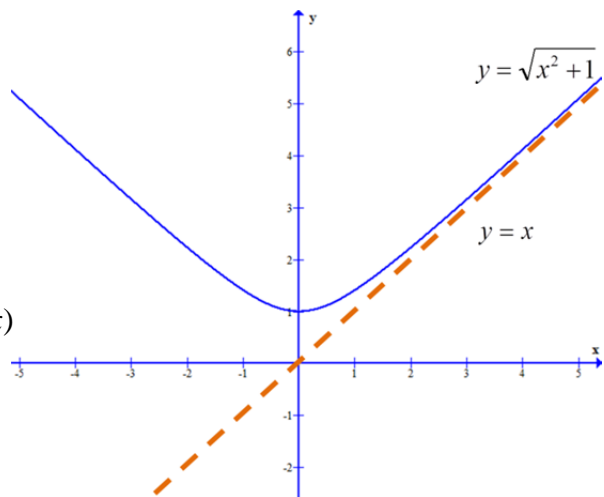


\*註記： $f(x)$ 在此處沒定義，  
因為根號裡面 $<0$ 。

**Example 3 :**  $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 1} - x) = ?$

**Solution :**

$$\begin{aligned} & \lim_{x \rightarrow \infty} (\sqrt{x^2 + 1} - x) \\ &= \lim_{x \rightarrow \infty} \frac{(\sqrt{x^2 + 1} + x)(\sqrt{x^2 + 1} - x)}{(\sqrt{x^2 + 1} + x)} \\ &= \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x^2 + 1} + x} \\ &= 0 \end{aligned}$$



\* 註記：

1. 這個例子  $y = x$  稱為  $y = \sqrt{x^2 + 1}$  的 slant(斜) asymptote。
2. 一般來說，若  $f(x) \neq ax + b$  when  $|x|$  is “large” ( $a \neq 0$ )，  
且  $\lim_{x \rightarrow \infty} [f(x) - (ax + b)] = 0$  or  $\lim_{x \rightarrow -\infty} [f(x) - (ax + b)] = 0$ ，  
則稱  $y = ax + b$  為  $y = f(x)$  的 slant asymptote(斜漸進線)。