§2-5 Continuity

Definition：

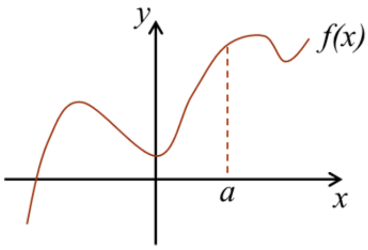
f is continuous at $x=a$

$\Leftrightarrow$ The graph of f has no break at $x=a$

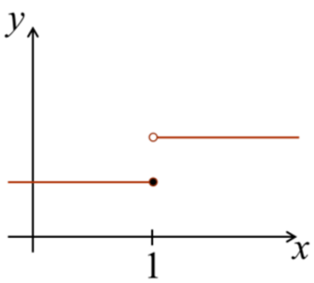
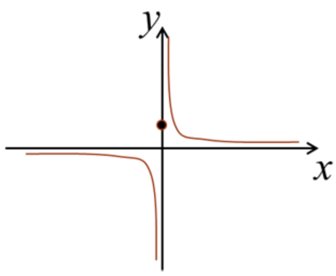
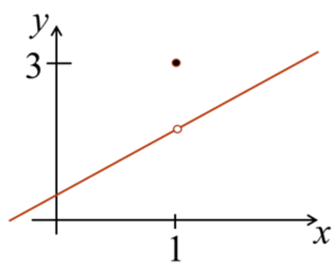
$\Leftrightarrow \lim_{x\to a} f(x) = f(\lim_{x\to a} x) = f(a)$ (極限和函數可交換)

$\Leftrightarrow$ Given $\varepsilon>0$, $\exists$ a $\delta=\delta_{\varepsilon}$ s.t.

if $|x-a|<\delta$, then $|f(x)-f(a)|<\varepsilon$.



∗ Three types of discontinuities：

1. **Jump**	2. **Infinite**	3. **Removable**
discontinuity at $x=1$.	$f(x)=\begin{cases}\frac{1}{x}, \text{if } x\neq 0\\ 1, \text{if } x=0\end{cases}$ discontinuity at $x=0$.	$f(x)=\begin{cases}\frac{x^2-1}{x-1}, x\neq 1\\ 3, x=1\end{cases}$ discontinuity at $x=1$.
		

Definition：

f is continuous on I.

$\Leftrightarrow f$ is continuous at every point in I.

＊Functions that are continuous on their domains：

Polynomials.

Exponentials & Log functions.

Trig. & Inverse trig. functions.

Rational.

Root functions.

Example 1：$\lim_{x\to 1}\sqrt{x} = ?$

Solution：$\lim_{x\to 1}\sqrt{x} = \sqrt{\lim_{x\to 1} x} = \sqrt{1} = 1$

Example 2：$\lim_{x\to 1}\sin^{-1}\left(\frac{1-\sqrt{x}}{1-x}\right) = ?$

Solution：$\lim_{x\to 1}\sin^{-1}\left(\frac{1}{1+\sqrt{x}}\right) = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$

Example 3：$f(x) = \begin{cases} \frac{x^2-x-2}{x-2}, & x \neq 2 \\ 1 & ,x = 2 \end{cases}$

(i) Is f continuous at x=2?

(ii) If not, classify the type of discontinuity of f at x=2.

Solution：

(i) $\lim_{x\to 2}\left(\frac{x^2-x-2}{x-2}\right) = \lim_{x\to 2}(x+1) = 3 \neq 1 \Rightarrow$ No!

(ii) Removable discontinuity.

Example 4： $f_1(x)=[x]$， $f_2(x)=\dfrac{x-7}{|x-7|}$

(i) Find, respectively, the set of discontinuities of f_i, i =1, 2.

(ii) Classify the type of discontinuity of f_i, i =1, 2.

Solution：

(i) f_1 ：discontinuous at all the integers.

f_2 ：discontinuous at x=7.

(ii) All are jump discontinuity.

Example 5：$f(x)=\begin{cases} cx^2+1, x\le 3. \\ cx-1\ , x>3. \end{cases}$ Find c so that f is continuous on (-∞,∞).

Example 6：$f(x)=\begin{cases} 0\ , x \text{ is irrational.} \\ 1, \quad x \text{ is rational.} \end{cases}$ Find the set of discontinuity of f.

Solution：

因為有理數與無理數在實數線上接稠密(dense)(任意兩個不同的實數中間，一定存在一個有理數和無理數)，因此f在所有實數點上皆不連續。

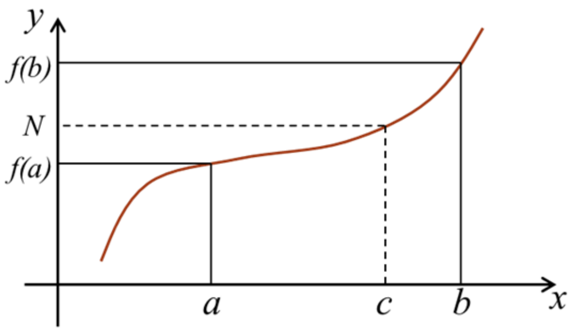
Example 7：$f(x)=\begin{cases} 0\ , x \text{ is irrational.} \\ x, \quad x \text{ is rational.} \end{cases}$ Find the set of discontinuity of f.

Solution： $\lim_{x\to 0} f(x)=0=f(0).\Rightarrow$在 0 點 f is continuous.

In fact, it is the only continuity.

⇒Discontinuity of f = R-{0}.

＊The Intermediate Value Theorem (中間值定理) (IVT)

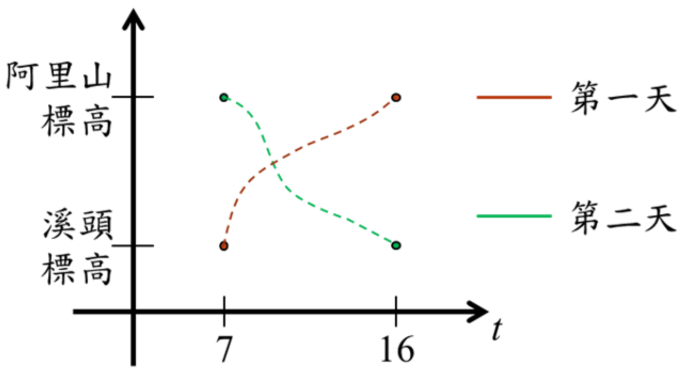


- f is cont. on $[a,b]$.
- $f(a) < N < f(b)$ or $f(b) < N < f(a)$

$\Rightarrow \exists\ c \in (a,b)$ s.t. $f(c) = N$

Example 8：小明早上 7 點從溪頭出發，一路往上，爬上阿里山，於下午 4 點抵達，在阿里山休息了一個晚上以後，隔天 7 點沿著相同路線往　下走回溪頭，一樣在下午 4 點抵達。試說明小明在這兩天所經過的路徑中，必然有一點小明在這兩天的同一時間經過。

Solution：



Let $f(t)$ = (第一天小明在 t 時間的標高) - (第二天小明在 t 時間的標高)

$\Rightarrow f(7) < 0$ and $f(16) > 0$

$\Rightarrow \exists\ t$ s.t. $f(t) = 0$

＊問：如果這條“溪－阿路線”不是一直往上(有上上下下的路段)，以上結論還是對嗎？