§10-2 Curves Defined by Parametric Equation

$x = f(t), \quad y = g(t)$

1. Tangents：

i. $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \text{if } \frac{dx}{dt} \neq 0$$

ii. $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

2. Areas：

$$A = \int_a^b y\,dx = \int_\alpha^\beta g(t) f'(t)\,dt .$$

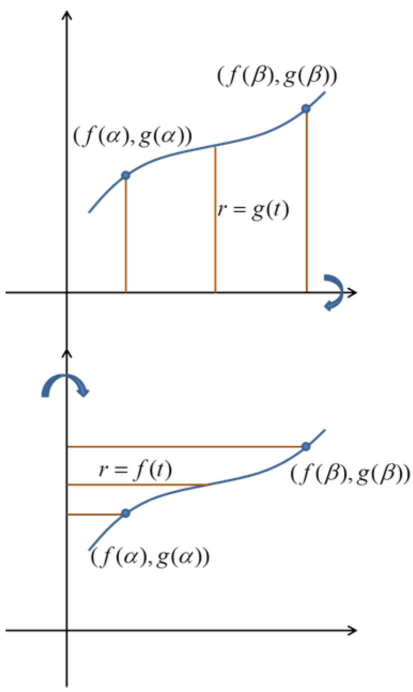
3. Arc Length：

$$L = \int_a^b ds = \int_a^b \sqrt{(dx)^2 + (dy)^2} = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt .$$

4. Surface Area：

$$S = \int_a^b 2\pi r\,ds$$

$$= \begin{cases} 2\pi \int_\alpha^\beta g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\ \\ 2\pi \int_\alpha^\beta f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \end{cases}$$



國立交通大學應用數學系 莊重教授

Example 1：

Given $x = t^2,\ y = t^3 - 3t,\ t \in R.$ Find the equations of the tangents to the above curve at (3 , 0).

Solution：

$$t^3 - 3t = 0$$
$$\Rightarrow t = 0\ (\text{不合})\ \text{or}\ \pm\sqrt{3}$$
$$\Rightarrow \frac{dy}{dx} = \frac{3t^2 - 3}{2t} = \frac{6}{2\left(\pm\sqrt{3}\right)} = \pm\sqrt{3}$$
$$\Rightarrow \text{Two tangents}: y = \sqrt{3}(x-3)\ \text{and}\ y = -\sqrt{3}(x-3).$$

Example 2：

Find the tangents to the cycloid $x = r(\theta - \sin\theta),\ y = r(1-\cos\theta)$ at $\theta = \frac{\pi}{3}$.

Solution：

$$\frac{dy}{dx} = \left.\frac{\sin\theta}{1-\cos\theta}\right|_{\theta=\frac{\pi}{3}} = \sqrt{3}$$
$$x = r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right),\ y = \frac{r}{2}$$
$$\Rightarrow \text{The tangent}: \sqrt{3}x - y = r\left(\frac{\pi}{\sqrt{3}} - 2\right).$$

Example 3：

Let $x = r(\theta - \sin\theta)$, $y = r(1-\cos\theta)$. Find $\dfrac{d^2 y}{dx^2}$.

Solution：

$$\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta}$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}$$

$$= \frac{\cos\theta\left(1-\cos\theta\right)-\sin^2\theta}{r\left(1-\cos\theta\right)^3}$$

$$= \frac{\cos\theta-1}{r\left(1-\cos\theta\right)^3}.$$

Example 4：

Find the area and arc length under one arch of the cycloid

$x = r(\theta - \sin\theta),\ y = r(1 - \cos\theta).\left(0 \le \theta \le 2\pi\right)$

Solution：

$$A = \int_0^{2\pi} r^2 \left(1 - \cos\theta\right)\left(1 - \cos\theta\right) d\theta$$

$$= r^2 \int_0^{2\pi} \ \ 1 - 2\cos\theta + \cos^2\theta d\theta = 3\pi r^2$$

$$L = r\int_0^{2\pi} \sqrt{\left(1 - \cos\theta\right)^2 + \sin^2\theta} d\theta$$

$$= r\int_0^{2\pi} \sqrt{2 - 2\cos\theta} d\theta$$

$$= 2r\int_0^{2\pi} \sin\frac{\theta}{2} d\theta$$

$$= 8r.$$

Example 5：

Show that the surface area of a sphere of radius r is $4\pi r^2$.

Solution：

$$S = \int 2\pi r ds = 2\pi \int_0^{\pi} r^2 \sin\theta d\theta = 4\pi r^2.$$