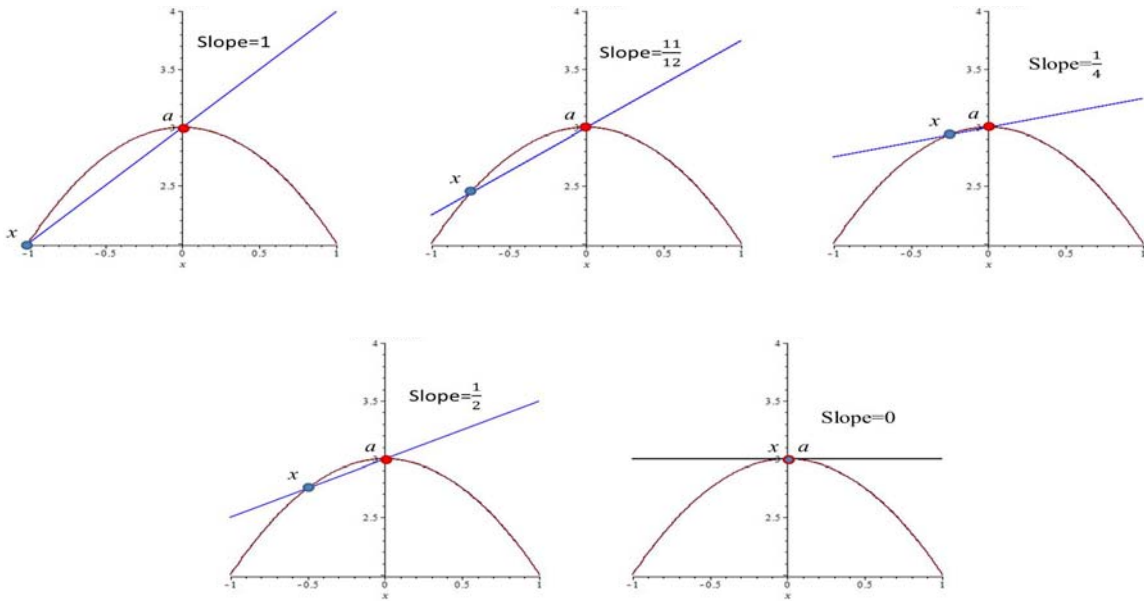


§2-2 – 2-3 Limit

* $f(x) = \frac{g(x) - g(a)}{x - a}$ 函數上過 x 點和 a 點的割線斜率

$\lim_{x \rightarrow a} f(x) = ? = g$ 在過 $x=a$ 點的切線斜率。



* $\lim_{x \rightarrow a} f(x) = L$, $\lim_{x \rightarrow a^+} f(x) = L$ (右極限), $\lim_{x \rightarrow a^-} f(x) = L$ (左極限)

$\xrightarrow{\hspace{1.5cm}}$ $\xleftarrow{\hspace{1.5cm}}$
 $\hspace{1.5cm} \downarrow$
 $\hspace{1.5cm} a$

- i. $x \rightarrow a$, x 愈來愈靠近 a , 但從來不是 a .
- ii. • 極限存在 \Rightarrow 極限唯一 \Rightarrow 左極限 = 右極限.
 • 左、右極限分別存在且相等 \Rightarrow 極限存在.
- iii. • L 不一定是 $f(a)$.

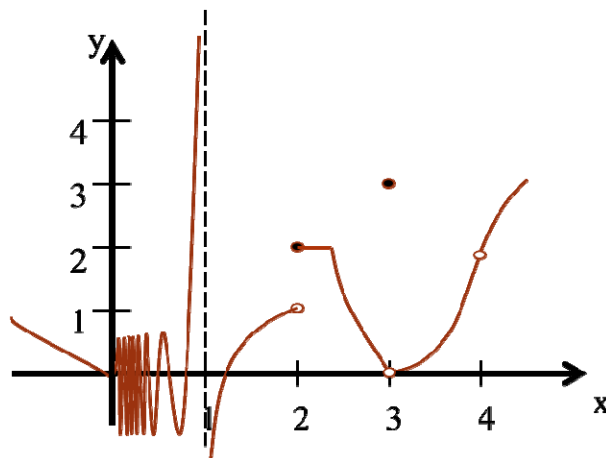
iv. If

$$f : \begin{cases} 1. \text{ Polynomials} \\ 2. \text{ Rational functions, } a \in \text{Domain}(f) \\ 3. \text{ The graph of } f \text{ has no break,} \end{cases}$$

$\Rightarrow L = f(a).$

*更一般來說，若 f : continuous at a , 則 $L=f(a)$

Example 1 :



(1) 那些 a 使得上述左、右極限，分別存在。

Solution : (i) $R - \{1\}$ (也即只有 $a = 1$ 時，左極限不存在).

(ii) $R - \{0, 1\}$ (也即在 $a = 0$ or 1 時，右極限不存在).

(2) 那些 a 使得上述極限存在。

Solution : $R - \{0, 1, 2\}$ (註)

*註-1：

$$\text{Let } x_n = \frac{1}{2n\pi + \theta}, \quad \theta \in R. \quad \text{Then } \lim_{n \rightarrow \infty} x_n = 0$$

$$\lim_{n \rightarrow \infty} \sin \frac{1}{x_n} = \lim_{x_n \rightarrow 0} \sin \frac{1}{x_n} = \lim_{x_n \rightarrow 0} \sin \theta = \sin \theta.$$

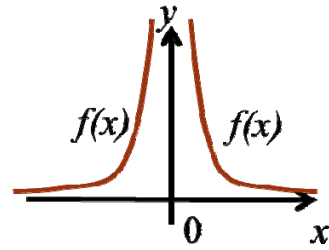
此表示當 x 由右邊趨近 0 時，若趨近路徑不相同時，極限不同。

*註-2：

$\lim_{x \rightarrow 1} f(x)$ 我們可以說極限不存在，

或寫成 $\lim_{x \rightarrow 1^+} f(x) = \infty$ 且 $\lim_{x \rightarrow 1^-} f(x) = -\infty$ ，

但不可寫成 $\lim_{x \rightarrow 1} f(x) = \infty$!



*註-3：

若如右圖，則可寫成 $\lim_{x \rightarrow 1} f(x) = \infty$

(3) 那些 a 使得 $L = f(a)$.

Solution : $R - \{0,1,2,3,4\}$

Example 2 : $\lim_{x \rightarrow 0} (x^2 + 1) = ?$

Solution : $\lim_{x \rightarrow 0} (x^2 + 1) = 1$

Example 3 : $\lim_{x \rightarrow 1} \frac{x-1}{x^2-1} = ?$

Solution : $\lim_{x \rightarrow 1} \frac{x-1}{x^2-1} = \lim_{x \rightarrow 1} \frac{1}{x+1} = \frac{1}{2}$

*註記：

1. 在極限的定義中， x 趨近於 1，但是 x 卻不能=1
2. 計算極限，不一定能直接帶進去算。

Example 4 : $\lim_{x \rightarrow 3^-} \frac{2x}{x-3} = ?$ $\lim_{x \rightarrow 3^+} \frac{2x}{x-3} = ?$

Solution : $\lim_{x \rightarrow 3^-} \frac{2x}{x-3} = \infty$ $\lim_{x \rightarrow 3^+} \frac{2x}{x-3} = -\infty$

Example 5 : $\lim_{x \rightarrow 1.5} \frac{2x^2 - 3x}{|2x - 3|} = ?$

Solution :

(i) 右極限： $\lim_{x \rightarrow 1.5^+} \frac{2x^2 - 3x}{|2x - 3|} = \frac{x(2x - 3)}{(2x - 3)} = x = 1.5$

(ii) 左極限： $\lim_{x \rightarrow 1.5^-} \frac{2x^2 - 3x}{|2x - 3|} = \frac{x(2x - 3)}{-(2x - 3)} = -x = -1.5$

$\Rightarrow \lim_{x \rightarrow 1.5} \frac{2x^2 - 3x}{|2x - 3|}$ 不存在

Example 6 : $\lim_{x \rightarrow 1} \left[\frac{1}{x-1} - \frac{2}{x^2-1} \right] = ?$

Solution : $\lim_{x \rightarrow 1} \left(\frac{x-1}{x^2-1} \right) = \lim_{x \rightarrow 1} \frac{1}{x+1} = \frac{1}{2}$

Example 7 : $\lim_{t \rightarrow 0} \frac{\sqrt{t^2+9}-3}{t^2} = ?$

Solution :

$$\begin{aligned} & \lim_{t \rightarrow 0} \frac{(\sqrt{t^2+9}-3)(\sqrt{t^2+9}+3)}{t^2(\sqrt{t^2+9}+3)} \\ &= \lim_{t \rightarrow 0} \frac{1}{\sqrt{t^2+9}+3} \\ &= \frac{1}{6} \end{aligned}$$

*註記：

本題目是要提醒大家，兩個分數相加減，應先通分，合併成一個分數。

*註記：

見根號可利用 $(a-b)(a+b) = a^2 - b^2$ 來化簡。

Example 8 : $\lim_{t \rightarrow 0} \left(\frac{1}{t\sqrt{1+t}} - \frac{1}{t} \right) = ?$

Solution :

合併： $\lim_{t \rightarrow 0} \left(\frac{1 - \sqrt{1+t}}{t\sqrt{1+t}} \right)$

$$\left(\frac{(a+b)(a-b)}{(a+b)} \right) := \lim_{t \rightarrow 0} \frac{(1 - \sqrt{1+t})(1 + \sqrt{1+t})}{t\sqrt{1+t}(1 + \sqrt{1+t})}$$

$$= \lim_{t \rightarrow 0} \left(\frac{-1}{\sqrt{1+t}(1 + \sqrt{1+t})} \right)$$

$$= \frac{-1}{2}$$

Example 9 : $\lim_{x \rightarrow 2} \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} = ?$

Solution :

$$\lim_{x \rightarrow 2} \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1}$$

$$= \lim_{x \rightarrow 2} \frac{(2-x)(\sqrt{3-x}+1)}{(2-x)(\sqrt{6-x}+2)}$$

$$= \frac{2}{4} = \frac{1}{2}$$

Example 10 : $\lim_{x \rightarrow 0} \frac{\sqrt[3]{1+cx}-1}{x} = ?$

Solution :

Let $a=1+cx$

$$\lim_{x \rightarrow 0} \frac{(a^{\frac{1}{3}} - 1)(a^{\frac{2}{3}} + a^{\frac{1}{3}} + 1)}{x(a^{\frac{2}{3}} + a^{\frac{1}{3}} + 1)} = \lim_{x \rightarrow 0} \frac{a-1}{x(a^{\frac{2}{3}} + a^{\frac{1}{3}} + 1)}$$

$$\lim_{x \rightarrow 0} \frac{cx}{x(a^{\frac{2}{3}} + a^{\frac{1}{3}} + 1)} = \lim_{x \rightarrow 0} \frac{c}{a^{\frac{2}{3}} + a^{\frac{1}{3}} + 1} = \frac{c}{3}$$

*註記：

本題同時運用
例題6與例題
7之技巧。

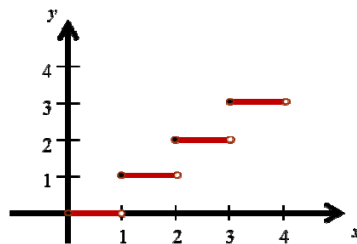
*註記：

利用乘法公式：

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2) \quad \text{去根號。}$$

Example 11 : $\lim_{x \rightarrow 2^+} [x] = ?$ $\lim_{x \rightarrow 2^-} [x] = ?$

Solution : $\lim_{x \rightarrow 2^+} [x] = 2$; $\lim_{x \rightarrow 2^-} [x] = 1$



Example 12 : $\lim_{x \rightarrow -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$ exists. Find a and its limit.

Solution :

$$\begin{aligned} & 3(-2)^2 + a(-2) + a + 3 = 0 \\ \Rightarrow & a = 15 \\ & \lim_{x \rightarrow -2} \frac{3x^2 + 15x + 18}{x^2 + x - 2} \\ & = \lim_{x \rightarrow -2} \frac{3(x+2)(x+3)}{(x+2)(x-1)} \\ & = -1 \end{aligned}$$

* 註記：
分母趨近於 0 當
 $x \rightarrow -2$ ，因此分子也
要趨近於 0 當
 $x \rightarrow -2$ ，才能保證極
限存在！

Example 13 : If $\lim_{x \rightarrow 0} \frac{f(x)}{x^2} = 5$, find the following limits.

(a) $\lim_{x \rightarrow 0} f(x)$ (b) $\lim_{x \rightarrow 0} \frac{f(x)}{x}$

Solution : (a) $\lim_{x \rightarrow 0} f(x) = 0$ (b) $\lim_{x \rightarrow 0} \frac{f(x)}{x} = 0$

Example 14 : $\lim_{x \rightarrow 4} f(x)$ exists. $f(x) = \begin{cases} \sqrt{x-4}, & x > 4 \\ 8-ax, & x < 4 \end{cases}$ Find a .

Solution : 右極限 = 0 = 左極限 = $8 - 4a$
 $\Rightarrow a = 2$

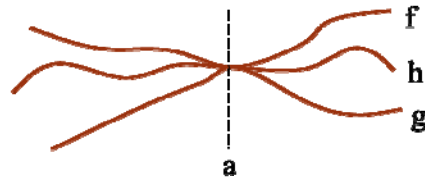
Limit Laws. $\lim_{x \rightarrow a} f(x)$, $\lim_{x \rightarrow a} g(x)$ exist.

$$\Rightarrow \lim_{x \rightarrow a} f(x) * g(x) = \lim_{x \rightarrow a} f(x) * \lim_{x \rightarrow a} g(x)$$

Here $*$ = +, -, \times .

若 $\lim_{x \rightarrow a} g(x)$ 存在且不為 0，則 $*$ 也可是 \div 。

The Sandwich Theorem.



If $f(x) \leq h(x) \leq g(x)$, except possibly at a , for x near a , and

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = L, \text{ then } \lim_{x \rightarrow a} h(x) = L.$$

Example 15 : $\lim_{x \rightarrow 0} x \sin \frac{1}{x} = ?$

Solution :

$$\begin{aligned} -|x| &\leq x \sin \frac{1}{x} \leq |x| \\ \Rightarrow \lim_{x \rightarrow 0} x \sin \frac{1}{x} &= 0 \end{aligned}$$

Example 16 : If $f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

prove that $\lim_{x \rightarrow 0} f(x) = 0$.

Example 17 : The figure shows a fixed circle c_1 with equation $(x-1)^2 + y^2 = 1$ and a shrinking circle c_2 with radius r and center the origin. P is the point $(0,r)$, Q is the upper point of intersection of the two circle, and R is the point of intersection of the line PQ and the x -axis. What happens to R as c_2 shrinks, that is as $r \rightarrow 0^+$?

