Chapter 4: Hilbert Spaces

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Outline

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   - Sobolev spaces

2 Projections
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   - Orthogonal projections

3 Riesz Representation Theorem

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   - Solving Poisson equations
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Definition

Let $X$ be a complex linear space. An inner product $(\cdot, \cdot)$ is a bilinear form: $X \times X \to \mathbb{C}$ which satisfies

(a) $(x, x) \geq 0$ and $(x, x) = 0$ if and only if $x = 0$,

(b) $(x, y) = (y, x)$,

(c) $(x, \alpha y + \beta z) = \alpha(x, y) + \beta(x, z)$.

The linear space $X$ equipped with the inner product $(\cdot, \cdot)$ is called an inner product space.
Examples.

- The space $\mathbb{C}^n$ with
  \[ (x, y) := \sum_i x_i y_i \]
  is an inner product space.

- Let $A$ be a symmetric positive definite matrix in $\mathbb{R}^n$. Define
  \[ \langle x, y \rangle_A := (x, Ay) \]
  Then $\langle \cdot, \cdot \rangle_A$ is an inner product in $\mathbb{R}^n$.

- The space $C[0, 1]$ with the inner product
  \[ (f, g) := \int_0^1 f(t)g(t) \, dt \]
  is an inner product space.
The space $L^2(0, 1)$ is the completion of $C[0, 1]$ with the above inner product. In fact, it is the space of all functions whose squares are Lebesgue integrable.

Let $\mathbb{T}$ be the unit circle and

$$L^2(\mathbb{T}) := \{ f : \mathbb{T} \to \mathbb{C} \mid \int_\mathbb{T} |f(t)|^2 dt < \infty \}$$

It is the space of all square summable and periodic functions.

The space $\ell^2(\mathbb{N})$ is defined to be

$$\ell^2(\mathbb{N}) := \{ x \mid x = (x_1, x_2, \cdots), \sum_{i=1}^{\infty} |x_i|^2 < \infty \}$$

equipped with the inner product

$$(x, y) := \sum_{i=1}^{\infty} x_i y_i.$$
Let $w_n > 0$ be a positive sequence. Define

$$\ell^2_w := \{ x | x : \mathbb{N} \to \mathbb{C}, \sum_{i=1}^{\infty} w_i |x_i|^2 < \infty \}$$

with the inner product:

$$(x, y) := \sum_{i=1}^{\infty} w_i x_i y_i$$

Let $w : (a, b) \to \mathbb{R}^+$ be a positive continuous function. Consider the space

$$L^2_w(a, b) := \{ f : (a, b) \to \mathbb{C} | \int_a^{b} |f(x)|^2 w(x) \, dx < \infty \}$$

equipped with the inner product

$$(f, g) := \int_a^{b} \overline{f(x)} g(x) w(x) \, dx.$$
We can define $\|x\| = \sqrt{(x, x)}$.

**Theorem**

Let $X$ be an inner product space. For any $x, y \in X$, we have

$$|(x, y)| \leq \|x\| \|y\|.$$ 

**Proof:**

1. From non-negativity of $(\cdot, \cdot)$, we get

$$0 \leq (x+ty, x+ty) = \|x\|^2 + 2\text{Re}(x, y)t + \|y\|^2 t^2 \text{ for all } t \in \mathbb{R}.$$ 

From this, we obtain one form of Cauchy-Schwarz:

$$|\text{Re}(x, y)|^2 \leq \|x\|^2 \|y\|^2.$$
2 We claim that

\[ |Re(x, y)| \leq \|x\| \|y\| \quad \text{for any } x, y \in X \]

if and only if

\[ |(x, y)| \leq \|x\| \|y\| \quad \text{for any } x, y \in X. \]

3 Suppose \((x, y)\) is not real, we choose a phase \(\phi\) such that \(e^{i\phi}(x, y)\) is real. Now we replace \(x\) by \(e^{i\phi}x\). Then

\[ |Re(e^{i\phi}x, y)| \leq \|x\| \|y\| \]

But the left-hand side is \(|(x, y)|\). This proves one direction. The other direction is trivial.
Cauchy-Schwarz and Triangle inequality

- From
  \[ \|x + y\|^2 = \|x\|^2 + 2\text{Re}(x, y) + \|y\|^2 \]

and

\[ (\|x\| + \|y\|)^2 = \|x\|^2 + 2\|x\| \|y\| + \|y\|^2, \]

by comparing the two equations, we get that the triangle inequality is equivalent to the Cauchy-Schwarz inequality.

- In fact, the following statements are equivalent:
  
  (a) For any \( x, y \in \mathcal{H} \), \( \text{Re}(x, y) \leq \|x\| \|y\| \);
  
  (b) For any \( x, y \in \mathcal{H} \), \( |(x, y)| \leq \|x\| \|y\| \);
  
  (c) For any \( x, y \in \mathcal{H} \), \( \|x + y\| \leq \|x\| + \|y\| \).
Remark

If we are care about the cosine law, that is

\[ \|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\| \cos \theta, \]

then we should define the angle between \( x \) and \( y \) by

\[ \cos \theta := \frac{\text{Re}(x, y)}{\|x\|\|y\|}. \]

However, this creates a problem, the orthogonality in this sense may not have \( (x, y) = 0 \). This is not what we want. So, we define the acute angle between two vectors \( x \) and \( y \) by

\[ \cos \theta := \left| \frac{(x, y)}{\|x\|\|y\|} \right|, \]

and we give up the traditional cosine law.
Proposition (Parallelogram law)

A normed linear space is an inner product space if and only if

\[ \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2, \text{ for all } x, y \in X. \]

Proof.

Suppose a norm satisfies the parallelogram law, we define

\[ (x, y) := \frac{1}{4} \left( \|x + y\|^2 - \|x - y\|^2 - i\|x + iy\|^2 + i\|x - iy\|^2 \right). \]

I leave you to check the parallelogram law implies the bilinearity of the inner product.
Sobolev spaces

1. The $H^1$ space
2. The $H^1_0$ space
3. Poincaré inequality
4. Optimal constant in Poincaré inequality
The $H^1$ space.

Similarly, we define

$$H^1(a, b) = \{ u : (a, b) \to \mathbb{C} \mid \int_a^b \left( |u(x)|^2 + |u'(x)|^2 \right) \, dx < \infty \}$$

with the inner product

$$(u, v) = \int_a^b (\overline{uv} + \overline{u'}v') \, dx.$$ 

Why is $H^1(a, b)$ complete? \(^1\) Indeed, it is the completion of $C^1[a, b]$, or $C^\infty[a, b]$ under the above inner product.

---

\(^1\) If $u_n \to u$ in $L^2(a, b)$ and $u'_n \to v$ in $L^2(a, b)$, then prove $u' = v$. 
The $H^1_0$ spaces

- Define $H^1_0(a, b) = \{ u \in H^1(a, b) | u(a) = u(b) = 0 \}$.
- $u(a)$ and $u(b)$ are well-defined for $u \in H^1(a, b)$. In fact, for any two points $x_1$ and $x_2$ near $a$, we can express

$$|u(x_2) - u(x_1)| = \left| \int_{x_1}^{x_2} u'(x) \, dx \right|$$

$$\leq \left( \int_{x_1}^{x_2} 1^2 \, dx \right)^{1/2} \left( \int_{x_1}^{x_2} |u'(x)|^2 \, dx \right)^{1/2}$$

$$\leq (x_2 - x_1)^{1/2} \|u'\| \to 0 \text{ as } x_1, x_2 \to a.$$

- Alternatively, $H^1_0(a, b)$ is the completion of $C_0^\infty[a, b]$ under the above inner product. Here, $C_0^\infty[a, b]$ are those $C^\infty$ function on $[a, b]$ satisfying zero boundary condition.
Poincaré inequality

- In $H^1_0(a, b)$, we can define another inner product
  \[ \langle u, v \rangle := \int_a^b u'(x)v'(x) \, dx. \]

- We check that $\langle u, u \rangle = 0$ implies $u \equiv 0$.
  From $\int_a^b |u'(x)|^2 \, dx = 0$, we get that $u'(x) \equiv 0$ on $(a, b)$.
  This together with $u(a) = u(b) = 0$ lead to $u \equiv 0$.

- Now, in $H^1_0$, we have two norms:
  \[ \|u\|_1^2 \equiv \|u\|^2 + \|u'\|^2, \quad \|u\|_2^2 \equiv \|u'\|^2. \]
  We claim that they are equivalent in $H^1_0$. 

We recall that two norms $\| \cdot \|_1, \| \cdot \|_2$ are equivalent in a normed space $X$ if there exist two positive constants $C_1, C_2$ such that for any $u \in X$, we have

$$C_1\|u\|_2 \leq \|u\|_1 \leq C_2\|u\|_2.$$ 

Clearly we have

$$\|u'\|^2 \leq \|u\|^2 + \|u'\|^2.$$ 

**Theorem (Poincaré inequality)**

There exists a constant $C' > 0$ such that for any $u \in H^1_0(a, b)$, we have

$$\|u\|^2 \leq C'\|u'\|^2. \quad (1.1)$$
Proof of Poincaré inequality

1. \[ u(x) = u(a) + \int_a^x u'(y) \, dy = \int_a^x u'(y) \, dy. \]

\[
|u(x)|^2 = \left| \int_a^x u'(y) \, dy \right|^2 \\
\leq \left( \int_a^x 1^2 \, dy \right) \left( \int_a^x |u'(y)|^2 \, dy \right) \\
\leq (x - a) \left( \int_a^b |u'(y)|^2 \, dy \right)
\]

2. We integrate \( x \) over \((a, b)\) to get

\[
\int_a^b |u(x)|^2 \, dx \leq \frac{(b - a)^2}{2} \int_a^b |u'(y)|^2 \, dy.
\]
Remarks

1. The Poincaré inequality is valid by just assuming $u(a) = 0$.
2. It is also valid by assuming $\int_a^b u(x) \, dx = 0$.
3. Dimension analysis for the Poincaré inequality: Denote the dimensions by $[x] = L$ and $[u] = U$. We have

$$[\|u\|] = (U^2 L)^{1/2} = U L^{1/2}, \quad [\|u'\|] = (U L^{-1}) L^{1/2} = U L^{-1/2}$$

4. Thus in $\|u\| \leq C' \|u'\|$, the dimension $[C'] = L$. 
Best constant in Poincaré inequality

To find the best constant $C$ in the Poincaré inequality, we look for the following minimum

$$\min_{u(a)=u(b)=0} \frac{\int_a^b u'(x)^2 \, dx}{\int_a^b u(x)^2 \, dx}$$

This problem is equivalent to

$$\min_{u(a)=u(b)=0} \int_a^b u'(x)^2 \, dx \quad \text{subject to} \quad \int_a^b u(x)^2 \, dx = 1.$$ 

By the method of Lagrange multiplier, there exists a $\lambda$ such that

$$\delta \left( \int_a^b u'(x)^2 \, dx - \lambda \int_a^b u(x)^2 \, dx \right) = 0.$$
The corresponding Euler-Lagrange equation is

\[-u'' - \lambda u = 0\]

with the two boundary condition \( u(a) = u(b) = 0 \).

This is a standard eigenvalue problem. The minimal value of \( \lambda \) is the first eigenvalue of \(-D^2\) with the Dirichlet boundary condition. The corresponding eigenvector and eigenvalue are

\[u(x) = \sin \left( \frac{x - a}{b - a} \pi \right), \quad \lambda = \left( \frac{\pi}{b - a} \right)^2.\]

Thus, the best constant is

\[C' = \frac{1}{\sqrt{\lambda}} = \frac{b - a}{\pi}.\]
Exercise: The weighted Sobolev space. Let \( w(x) > 0 \) on \([a, b]\).
Define the inner product

\[
\langle u, v \rangle_w := \int_a^b u'(x)v'(x)w(x) \, dx
\]

and the corresponding norm \( \|u'\|^2_w := \langle u, u \rangle_w \). Let

\[
H^1_{w,0}(a, b) := \{ u : (a, b) \to \mathbb{C} \mid \|u'\|_w < \infty, u(a) = u(b) = 0 \}
\]

Then the space \( H^1_{0,w}(a, b) = H^1_0 \) and the norm \( \|u'\|_w \) is equivalent to \( \|u'\| \).
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Projections in Banach spaces

Definition

(a) A projection \( P \) in a Banach space \( X \) is a linear mapping from \( X \) to \( X \) satisfying \( P^2 = P \).

(b) The direct sum of two subspaces \( \mathcal{M} \) and \( \mathcal{N} \) in a Banach space \( X \) is defined to be

\[
\mathcal{M} \oplus \mathcal{N} := \{ x + y \mid x \in \mathcal{M}, \ y \in \mathcal{N} \}.
\]
Theorem

If $P$ is a projection on a linear space $X$, then

$X = \text{Ran} P \oplus \text{Ker} P$, and $\text{Ran} P \cap \text{Ker} P = \{0\}$.

Conversely, if $X = M \oplus N$ and $M \cap N = \{0\}$, then any $x \in X$ can be uniquely represented as $x = y + z$ with $y \in M$ and $z \in N$. Furthermore, the mapping $P : x \mapsto y$ is a projection.

- Here, $\text{Ran} P$: range of $P$, $\text{Ker} P$: kernel of $P$. 
Proof

$(\Rightarrow)$

1. We first show that $x \in \text{Ran } P \iff x = Px$. $(\Leftarrow)$ If $x = Px$ clearly $x \in \text{Ran } P$. $(\Rightarrow)$ If $x \in \text{Ran } P$, then $x = Py$ for some $y \in \mathcal{H}$. From $P^2 = P$, we get $Px = P^2 y = Py = x$.

2. Next, if $x \in \text{Ran } P \cap \text{Ker } P$, then $x = Px = 0$. Hence, \( \text{Ran } P \cap \text{Ker } P = \{0\} \).

3. Finally, we can decompose $x \in X$ into

$$x = Px + (x - Px).$$

The part $Px \in \text{Ran } P$. The other part $x - Px \in \text{Ker } P$ because $P(x - Px) = Px - P^2 x = 0$. 

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If \( x = y_1 + z_1 = y_2 + z_2 \) with \( y_i \in \mathcal{M} \) and \( z_i \in \mathcal{N} \), then
\[ y_1 - y_2 = z_2 - z_1 \]
and it is in \( \mathcal{M} \cap \mathcal{N} \). Thus, \( y_1 = y_2 \) and \( z_1 = z_2 \).

For \( y \in \mathcal{M} \), \( Py = y \). For any \( x \), \( Px \in \mathcal{M} \), hence
\[ P(Px) = Px. \]

Remarks.

1. If \( P \) is a projection, so is \( I - P \).
2. We have \( \text{Ran} P = \text{Ker}(I - P) \), \( \text{Ker} P = \text{Ran}(I - P) \).
3. A projection in a Banach space needs not be continuous in general.
**Theorem**

Let $X$ be a Banach space and $P$ is a projection in $X$.

(a) If $P$ is continuous, then both $\text{Ker} P$ and $\text{Ran} P$ are closed.

(b) On the other hand, if $Y$ is a closed subspace and there exists a closed subspace $Z$ such that $X = Y \oplus Z$. Then the projection $P : x \mapsto y$ is continuous, where $x = y + z$ is the decomposition of $x$ with $y \in Y$ and $z \in Z$. 
Proof

1. We show the graph of $P$ is closed. That is, if $x_n \to x$ and $y_n := Px_n \to y$, then $y \in Y$ and $x - y \in Z$ (thus, $Px = y$). From the decomposition, we have $y_n \in Y$ and $z_n := x_n - y_n \in Z$. From the closeness of $Y$, we get $y \in Y$. From $x_n \to x$ and $y_n \to y$, we get $x_n - y_n$ converges to $x - y$. From the closeness of $Z$, we get $x - y \in Z$. Thus, $x = y + (x - y)$ with $y \in Y$ and $x - y \in Z$.

2. The theorem follows from the closed graph theorem: A closed graph linear map $A$ from Banach space $X$ to Banach space $Y$ is also continuous.
Orthogonal projections in Hilbert spaces

Theorem (Orthogonal Projection Theorem)

Let \( \mathcal{H} \) be a Hilbert space and let \( \mathcal{M} \subset \mathcal{H} \) be a closed linear subspace of \( \mathcal{H} \). Then

(a) for any \( x \in \mathcal{H} \), there exists a unique \( y \in \mathcal{M} \) such that

\[
\|x - y\| = \min_{z \in \mathcal{M}} \|x - z\|;
\]

(b) \( (x - y) \perp \mathcal{M} \);

(c) the mapping \( P : x \mapsto y \) is a projection.
Proof

Existence:

1. Let \( \ell = \inf_{z \in M} \| x - z \|^2 \). and let \( \{y_n\} \) be a minimal sequence of \( \| x - \cdot \|^2 \) in \( M \). That is \( y_n \in M \) and 
\[
\lim_{n \to \infty} \| y_n - x \|^2 = \ell.
\]

2. Then from the parallelogram law
\[
\frac{1}{2} \| y_m - y_n \|^2 = \| y_m - x \|^2 + \| y_n - x \|^2 - 2 \| \frac{y_m + y_n}{2} - x \|^2.
\]

The first two terms tend to \( 2\ell \) as \( n, m \to \infty \), while the last term is greater than \( 2\ell \) by the definition of \( \ell \). This implies \( \{y_n\} \) is a Cauchy sequence in \( M \) hence it has a limit \( y \) in \( M \).
Figure: Orthogonal projection of $x$ onto a closed subspace $\mathcal{M}$. 

$\{y_n\}$ are minimal sequence.
Uniqueness:

- Suppose \( y_1 \) and \( y_2 \) are two minima, that is \( \|y_i - x\|^2 = \ell \).
  
  By the parallelogram law,
  \[
  \frac{1}{2}\|y_1 - y_2\|^2 = \|y_1 - x\|^2 + \|y_2 - x\|^2 - 2\left\| \frac{y_1 + y_2}{2} - x \right\|^2 \leq 2\ell - 2\ell = 0.
  \]

Orthogonality: \( (x - y) \perp \mathcal{M} \).

- From
  \[
  \|x - y\|^2 \leq \|x - y - tz\|^2 = \|x - y\|^2 - 2\text{Re}(x - y, tz) + |t|^2\|z\|^2
  \]
  
  for all \( t \in \mathbb{C} \) and \( z \in \mathcal{M} \), we choose \( t = \epsilon e^{i\phi} \) so that
  \[
  \text{Re}(x - y, tz) = |t|\|(x - y, z)\|
  \]

Then we get
\[
\epsilon |(x - y, z)| \leq \epsilon^2 |z|^2.
\]

Taking \( \epsilon \to 0^+ \), we get \( (x - y, z) = 0 \).
Orthogonal Projection

- Let $\mathcal{H}$ be a Hilbert space. $\mathcal{M} \subset \mathcal{H}$ be a subset. Define the orthogonal complement of $\mathcal{M}$ by

$$\mathcal{M}^\perp := \{ x \in \mathcal{H} | x \perp y \text{ for all } y \in \mathcal{M} \}.$$  

- The orthogonal complement of a subset $\mathcal{M}$ in $\mathcal{H}$ is a closed linear subspace.
Corollary

If \( \mathcal{M} \) is a closed subspace of a Hilbert space \( \mathcal{H} \), then \( \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp \) and \( (\mathcal{M}^\perp)^\perp = \mathcal{M} \).

Proof.

We only prove \( (\mathcal{M}^\perp)^\perp \subset \mathcal{M} \). Suppose \( x \in (\mathcal{M}^\perp)^\perp \). That is, \( (x, w) = 0 \) for all \( w \in \mathcal{M}^\perp \). By the orthogonal projection theorem, we can decompose \( x = y + z \) with \( y \in \mathcal{M} \) and \( z \in \mathcal{M}^\perp \). Then \( 0 = (x, w) = (y + z, w) = (z, w) \) for all \( w \in \mathcal{M}^\perp \). Since \( z \in \mathcal{M}^\perp \), we can take \( w = z \) and get \( (z, z) = 0 \). Hence, \( x = y \in \mathcal{M} \). This proves \( (\mathcal{M}^\perp)^\perp \subset \mathcal{M} \).
**Theorem**

Let $P : \mathcal{H} \to \mathcal{H}$ be a projection. The following two statements are equivalent:

(a) $(P x_1, x_2) = (x_1, P x_2)$ for all $x_1, x_2 \in \mathcal{H}$;

(b) $\mathcal{H} = \text{Ran } P \oplus \text{Ker } P$ and $\text{Ran } P \perp \text{Ker } P$.

**Definition**

A mapping $P$ on $\mathcal{H}$ is called an orthogonal projection if it satisfies (i) $P^2 = P$, (ii) $(P x_1, x_2) = (x_1, P x_2)$. 


Proof

1. (a) ⇒ (b): For any \( x \in \text{Ran}P \), then \( x = Py \) for some \( y \in \mathcal{H} \). Then for any \( z \in \text{Ker} P \),

\[
(x, z) = (Py, z) = (y, Pz) = 0.
\]

Hence, \( \text{Ran}P \perp \text{Ker} P \).

2. (b) ⇒ (a): For any \( x_1, x_2 \in \mathcal{H} \), they can be uniquely decomposed into

\[
x_1 = y_1 + z_1, \hspace{1cm} x_2 = y_2 + z_2, \hspace{1cm} \text{with} \hspace{0.5cm} y_i \in \mathcal{M}, z_i \mathcal{M}^\perp.
\]

Thus,

\[
(Px_1, x_2) = (y_1, y_2) = (x_1, Px_2).
\]
Examples

**Example 1.** Given a vector $y \in \mathcal{H}$. Define $P : x \mapsto (y, x) \frac{y}{\|y\|^2}$. Then $\text{Ran} P = \langle \{y\} \rangle$ and $\ker P = y^\perp$.

**Example 2.**

1. Given $n$ independent vectors $\{v_1, \cdots, v_n\}$ in $\mathcal{H}$. Let $\mathcal{M} = \langle \{v_1, \cdots, v_n\} \rangle$.
   
   Given any $x \in \mathcal{H}$, the orthogonal projection $y$ of $x$ on $\mathcal{M}$ satisfies:
   
   $$ y = \arg \min \left\{ \frac{1}{2} \|x - z\|^2 \mid z \in \mathcal{M} \right\}. $$

2. The Euler-Lagrange equation is $(x - y) \perp \mathcal{M}$. 
3 Since $y \in \mathcal{M}$, we can express $y$ as $y = \sum_{i=1}^{n} \alpha_i v_i$. The condition $(x - y) \perp \mathcal{M}$ is equivalent to $(x - y, v_i) = 0$, $i = 1, \cdots, n$. This leads to the following $n \times n$ system of linear equations

$$\sum_{j=1}^{n} (v_i, v_j) \alpha_j = (x, v_i), i = 1, \cdots, n.$$  

From the independence of $\{v_1, \cdots, v_n\}$, we can get a unique solution of this equation.
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Dual space

- Let \( \mathcal{H} \) be a normed linear space. The set

\[
\mathcal{H}^* := \{ \ell : \mathcal{H} \to \mathbb{C} \text{ bounded linear functional} \}
\]

forms a linear space called the dual space of \( \mathcal{H} \). It is a Banach space equipped with the operator norm.

- Given a \( y \in \mathcal{H} \), the mapping \( \ell_y(x) := (y, x) \) is a bounded linear functional, by Cauchy-Schwarz inequality. Its norm \( \|\ell_y\| \leq \|y\| \). On the other hand, by choosing \( x = y/\|y\| \), we obtain \( \|\ell_y\| \geq |\ell_y(y/\|y\|)| = \|y\| \). Thus \( \|\ell_y\| = \|y\| \).

- Riesz representation theorem: every bounded linear functional on \( \mathcal{H} \) must be in this form. In other word, \( \mathcal{H}^* \) is isometric to \( \mathcal{H} \).
Theorem (Riesz representation theorem)

Let \( \ell \) be a bounded linear functional on a Hilbert space \( \mathcal{H} \). Then there exists a unique \( y \in \mathcal{H} \) such that \( \ell(x) = (y, x) \).

Proof.

1. We suppose \( \ell \neq 0 \). Our goal is to find \( y \) such that \( \ell(x) = (y, x) \). We first notice that such \( y \) must be in \((\text{Ker } \ell)^\perp\) and \( P : x \mapsto (y, x) y/\|y\|^2 \) is an orthogonal projection.

2. Let \( \mathcal{N} = \text{Ker } \ell \). Then \( \mathcal{N} \) is closed and \( \mathcal{N} \neq \mathcal{H} \). Hence there exists a \( z_1 \notin \mathcal{N} \).
3 By the orthogonal projection theorem, there exists a \( y_1 \in \mathcal{N} \) and \( z := (z_1 - y_1) \perp \mathcal{N} \). From \( z_1 \notin \mathcal{N} \), we get \( z \neq 0 \).

4 Let

\[
 Px := \frac{\ell(x)}{\ell(z)} z.
\]

Then \( P \) is an orthogonal projection, i.e. \( P^2 = P \) and \( \mathcal{H} = \text{Ran} P \oplus \text{Ker} P \), and \( \text{Ran} P \perp \text{Ker} P \). Since \( \text{Ran} P = \{ \alpha z | \alpha \in \mathbb{C} \} \) and \( \text{Ker} P = \text{Ker} \ell = \mathcal{N} \). We thus have

\[
 \mathcal{H} = \{ \alpha z | \alpha \in \mathbb{C} \} \oplus \text{Ker} \ell.
\]

5 Thus, any \( x \in \mathcal{H} \) can be represented uniquely by

\[
 x = \alpha z + m, \quad m \in \mathcal{N}, \quad \alpha = (z, x)/\|z\|^2.
\]
6 We have

\[ \ell(x) = \ell(\alpha z) = \frac{1}{\|z\|^2} (z, x) \ell(z) = (y, x). \]

where, \( y := \frac{\ell(z)}{\|z\|^2} z. \) We have shown the existence of \( y \) such that \( \ell(x) = (y, x). \)

7 For the uniqueness, suppose there are \( y_1 \) and \( y_2 \) such that \( \ell_{y_1} = \ell_{y_2}. \) That is,

\[ (y_1, x) = (y_2, x), \text{ for all } x \in \mathcal{H}. \]

Choose \( x = y_1 - y_2, \) we obtain \( \|y_1 - y_2\| = 0. \)
Outline

1. Hilbert spaces, Basic
   - Inner product structure
   - Sobolev spaces
2. Projections
   - Projections in Banach spaces
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3. Riesz Representation Theorem
4. Application of Riesz representation theorem
   - Solving Poisson equations
   - Error estimates for finite element method
Applications: Solving Poisson equation

We consider the Poisson equation on a bounded domain $\Omega \subset \mathbb{R}^n$:

\[(P) : \triangle u = f \text{ in } \Omega, u = 0 \text{ on } \partial \Omega.\]

This problem can be reformulated as the following weak form:

\[(WP) : \text{Find } u \in H^1_0(\Omega) \text{ such that } (\nabla u, \nabla v) = -(f, v) \text{, for all } v \in C^1_0.\]

**Theorem**

Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. Let $f \in L^2(\Omega)$. Then $(WP)$ has a unique solution in $H^1_0(\Omega)$. 
To show the existence for the Poisson equation, we need the following lemma. It was proven before.

**Lemma (Poincaré inequality)**

Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. Then there exists a constant $C$ such that for $u \in H_0^1(\Omega)$, we have

$$\|u\|_2 \leq C\|\nabla u\|_2.$$
From the Poincaré’s inequality, we see that
\[
\langle u, v \rangle_1 := (\nabla u, \nabla v) := \int_{\Omega} \nabla u(x) \cdot \nabla v(x) \, dx
\]
defines an inner product in \( H^1_0(\Omega) \).

On the other hand, for \( f \in L^2(\Omega) \), \( \ell_v := (f, v) \) is a bounded linear map in both \( L^2 \) and \( H^1_0 \):
\[
|\langle f, v \rangle| \leq \| f \| \| v \| \leq C \| f \| \| \nabla v \|
\]

Thus, by the Riesz representation theorem, there exists a unique \( u \in H^1_0(\Omega) \) such that
\[
\langle u, v \rangle_1 = (\nabla u, \nabla v) = (-f, v)
\]
for all \( v \in H^1_0(\Omega) \).
We consider the Poisson equation in one dimension:

$$-u'' = f \text{ on } (a, b), \quad u(a) = u(b) = 0.$$

We shall find an approximate solution by finite element method.

1. We choose an $n > 0$. Let $h := (b - a)/n$ the mesh size, $x_i = a + ih, \ i = 0, \ldots, n$ the grid point.
2. Define the nodal function $\phi_i(x)$ to be $\phi_i(x_j) = \delta_{ij}$ and $\phi(x)$ is continuous and piecewise linear.
3. Let

$$V_h = \langle \phi_1, \ldots, \phi_{n-1} \rangle$$

called the finite element space.
4 An element $v \in V_h$ is a continuous and piecewise linear function and is uniquely expressed by

$$v(x) = \sum_{i=1}^{n-1} v(x_i) \phi_i(x).$$

5 The approximate solution $u_h \in V_h$ is expressed as

$$u_h(x) = \sum_{i=1}^{n-1} U_i \phi_i(x).$$

6 We project the equation (4.2) onto $V_h$:

$$(-u'' - f, v) = 0, \text{ for all } v \in V_h.$$
This leads to the following equations for 
\[ U = (U_1, \cdots, U_{n-1})^T: \]
\[ \langle u_h, \phi_i \rangle_1 = (f, \phi_i), \ i = 1, \cdots, n - 1. \]

Or
\[
\sum_{j=1}^{n-1} (\phi_i', \phi_j') U_j = (f, \phi_i), \ i = 1, \cdots, n - 1.
\]

We can compute \((\phi_i, \phi_j)\) directly and obtain the matrix 
\[ A = (\phi_i', \phi_j')_{(n-1) \times (n-1)} \text{ as} \]
\[ A = \frac{1}{h} \text{diag}(-1, 2, -1) \]

This matrix is invertible.
Error of the approximate solution $u_h$

- Let $u$ be the exact solution and $e_h := u - u_h$ be the true error.
- Since both $u$ and $u_h$ satisfy
  \[(u', v') = (f, v), \quad (u'_h, v') = (f, v)\]
  for all $v \in V_h$,

  we obtain
  \[(e'_h, v') = 0\]
  for all $v \in V_h$.

  That is, $(u - u_h) \perp_1 V_h$. This is equivalent to say that $u_h$ is the \(\langle \cdot, \cdot \rangle_1\)-orthogonal projection of $u$ on $V_h$. 
Thus,

$$\|u' - u'_h\|_2 \leq \|u' - v'\|_2 \text{ for all } v \in V_h.$$ 

In particular, we can choose

$$v = \pi_h u := \sum_{i=1}^{n-1} u(x_i) \phi_i,$$

then

$$\|u' - u'_h\|_2 \leq \|u' - (\pi_h u)'\|_2. \quad (4.3)$$

Thus, the true error is controlled by the approximation error.
Approximation error

- The approximation error \( w(x) = u(x) - \pi_h u(x) \) satisfies
  \[ w(x_i) = w(x_{i+1}) = 0. \]

- We want to estimate \( \|w\|_2 \) and \( \|w'\|_2 \) in terms of \( \|u''\|_\infty \) and \( \|u''\|_2 \).
Approximation errors in terms of \( \|u''\|_{\infty} \):

\[
\|u - \pi_h u\|_2 \leq \sqrt{b - a} \frac{h^2}{8} \|u''\|_{\infty}
\]

From \( w(x_i) = w(x_{i+1}) = 0 \), we get for \( x \in (x_i, x_{i+1}) \), \( \exists \xi_i \)

\[
w(x) = \frac{w''(\xi_i)}{2} (x - x_i)(x - x_{i+1}).
\]

Hence

\[
\int_{a}^{b} |w(x)|^2 \, dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} |w(x)|^2 \, dx
\]

\[
\leq (b - a) \left( \frac{h^2}{8} \right)^2 \max_{x \in [a,b]} |w''(x)|^2.
\]

From \( w''(x) = u''(x) \) on \( (x_i, x_{i+1}) \), we get

\[
\|u - \pi_h u\|_2 \leq \sqrt{b - a} \frac{h^2}{8} \|u''\|_{\infty}
\]
\[ \| u' - (\pi_h u)' \|_2 \leq \sqrt{b - ah} \| u'' \|_{\infty} \]

- First, there exists a \( \zeta_1 \in (x_i, x_{i+1}) \) such that 
  \[ u'(\zeta_1) = \frac{(u(x_{i+1}) - u(x_i))}{h}. \]

- For any \( x \in (x_i, x_{i+1}) \), there exists \( \zeta_2 \in (x_i, x_{i+1}) \) such that 
  \[ u'(x) - u'(\zeta_1) = u''(\zeta_2)(x - \zeta_1). \]

- Therefore, we get for \( x \in (x_i, x_{i+1}) \)
  \[ u'(x) - (\pi_h u)'(x) = u'(x) - \frac{u(x_{i+1}) - u(x_i)}{h} = u''(\zeta_2)(x - \zeta_1). \]

\[
\int_{a}^{b} |u' - (\pi_h u)'|^2 \, dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} |u' - (\pi_h u)'|^2 \, dx \\
\leq \sum_{i=0}^{n-1} h^2 \max_{x \in [a, b]} |u''(x)|^2 = (b - a) h^2 \| u'' \|_{\infty}^2
\]
Approximation error in terms of $\|u''\|_2$

- It is desirable to estimate $\|u - \pi_h u\|_2$ in terms of $\|u''\|_2$.
- We should use the integral representation for the interpolation error $w$.
- Recall that for $w(x_i) = w(x_{i+1}) = 0$, $w$ has the representation:
  \[
  w(x) = h^2 \int_{x_i}^{x_{i+1}} g \left( \frac{x - x_i}{h}, \frac{y - x_i}{h} \right) w''(y) \, dy
  \]
  \[
  w'(x) = h \int_{x_i}^{x_{i+1}} g_x \left( \frac{x - x_i}{h}, \frac{y - x_i}{h} \right) w''(y) \, dy
  \]
  where $g$ is the Green’s function of $d^2/dx^2$ on $(x_i, x_{i+1})$.
  Thus, we can estimate $\|w\|_2$ and $\|w'\|_2$ in terms of $\|w''\|_2$ on $(x_i, x_{i+1})$. 

For \( x \in (x_i, x_{i+1}) \),

\[
|w(x)|^2 \leq h^4 \left( \int_{x_i}^{x_{i+1}} |g \left( \frac{x-x_i}{h}, \frac{y-x_i}{h} \right)|^2 \, dy \right) \left( \int_{x_i}^{x_{i+1}} |w''(y)|^2 \, dy \right).
\]

\[
\int_{x_i}^{x_{i+1}} |w(x)|^2 \, dx \leq h^4 \int_{x_i}^{x_{i+1}} \int_{x_i}^{x_{i+1}} |g \left( \frac{x-x_i}{h}, \frac{y-x_i}{h} \right)|^2 \, dy \, dx \left( \int_{x_i}^{x_{i+1}} |w''(y)|^2 \, dy \right)
\]

\[
\leq \frac{1}{90} h^4 \int_{x_i}^{x_{i+1}} |w''(y)|^2 \, dy.
\]

As we sum over \( i = 1, \ldots, n - 1 \), we get

\[
\|w\|_2 \leq \frac{1}{\sqrt{90}} h^2 \|w''\|_2.
\]

Similarly, we get

\[
\|w'\|_2 \leq \frac{1}{\sqrt{6}} h \|w''\|_2.
\]
Theorem

For $u \in H^2(a, b) \cap H_0^1[a, b]$, the interpolation error has the following estimates

$$\|u - \pi_h u\|_2 \leq \frac{1}{\sqrt{90}} h^2 \|u''\|_2,$$

$$\|u' - (\pi_h u)'\|_2 \leq \frac{1}{\sqrt{6}} h \|u''\|_2.$$
True error of the finite element method

**Theorem**

For the finite element method for problem (4.2), the true error $u - u_h$ has the following estimate

$$
\| u' - u'_h \|_2 \leq \frac{1}{\sqrt{6}} h \| u'' \|_2,
$$

$$
\| u - u_h \|_2 \leq \frac{1}{6} h^2 \| u'' \|_2.
$$
Proof by duality argument

1. Let $e_h = u - u_h$. We find the function $\phi_h$ such that $\phi_h'' = -e_h$ and $\phi(a) = \phi(b) = 0$.

2. Then

$$(e_h, e_h) = -(e_h, \phi_h'') = (e_h', \phi'_h) = (e_h', \phi'_h - (\pi_h \phi_h)'').$$

Here, I have used $(e'_h, v') = 0$ for all $v \in \text{Ran}(\pi_h)$.

3. Applying interpolation estimate to $\phi_h$, we get

$$\|e_h\|^2 \leq \|e'_h\| \|(\phi_h - \pi_h \phi_h)''\| \leq \frac{h}{\sqrt{6}} \|e'_h\| \|\phi''\| = \frac{h}{\sqrt{6}} \|e'_h\| \|e_h\|.$$ 

Hence, we get

$$\|e_h\|_2 \leq \frac{1}{\sqrt{6}} h \|e'_h\|_2 \leq \frac{1}{6} h^2 \|u''\|_2.$$